Line Integrals
MATH 311, *Calculus III*

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Motivation

Suppose we wish to find the mass of a piece of wire bent into the shape of a curve $C$.

If the density of the wire at $(x, y, z)$ is given by $\rho(x, y, z)$, then in a section of the wire of length $\Delta s$ the mass is approximately

$$\Delta m = \rho(x, y, z) \Delta s.$$ 

The mass of the wire is

$$m \approx \sum_{i=1}^{n} \rho(x_i, y_i, z_i) \Delta s_i.$$ 

The exact mass of the wire is

$$m = \lim_{\|P\| \to 0} \sum_{i=1}^{n} \rho(x_i, y_i, z_i) \Delta s_i$$

provided the limit exists and is the same for every choice of evaluation points.
We may adapt this approach to find the integral of other functions defined along curve $C$. 

Definition

The line integral of $f(x, y, z)$ with respect to arc length along the oriented curve $C$ in three-dimensional space is

$$\int_C f(x, y, z) \, ds = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i) \Delta s_i$$

provided the limit exists and is the same for every choice of evaluation points.
Line Integral

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provided the limit exists and is the same for every choice of evaluation points.
Theorem (Evaluation Theorem)

Suppose that $f(x, y, z)$ is continuous in a region $D$ containing the curve $C$ and that $C$ is described parametrically by $(x(t), y(t), z(t))$ for $a \leq t \leq b$, where $x(t)$, $y(t)$, and $z(t)$ have continuous first derivatives. Then

$$\int_C f(x, y, z) \, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} \, dt.$$
Theorem (Evaluation Theorem)

Suppose that \( f(x, y) \) is continuous in a region \( D \) containing the curve \( C \) and that \( C \) is described parametrically by \( (x(t), y(t)) \) for \( a \leq t \leq b \), where \( x(t) \) and \( y(t) \) have continuous first derivatives. Then

\[
\int_C f(x, y) \, ds = \int_a^b f(x(t), y(t)) \sqrt{[x'(t)]^2 + [y'(t)]^2} \, dt.
\]
Example

Evaluate the line integral

\[ \int_C 2x \, ds \]

where \( C \) is the portion of the parabola \( y = x^2 \) with endpoints \((0, 0)\) and \((1, 1)\).

**Remark:** we must describe the parabola parametrically.
Solution

\[ x(t) = t \]
\[ y(t) = t^2 \]

for \( 0 \leq t \leq 1 \).

\[ \int_C 2x \, ds = \int_0^1 2t \sqrt{(1)^2 + (2t)^2} \, dt \]
\[ = \int_0^1 2t \sqrt{1 + 4t^2} \, dt \]
\[ = \frac{1}{4} \int_1^5 u^{1/2} \, du \]
\[ = \frac{1}{6} \left( 5\sqrt{5} - 1 \right) \]
Example

A wire takes the shape of a semicircle $x^2 + y^2 = 1$ with $y \geq 0$. The linear density of the wire at any point is proportional to distance of the point from the line $y = 1$. Find the mass of the wire.
Solution

\[ \rho(x, y) = k(1 - y) \quad \text{(density of wire)} \]
\[ x(t) = \cos t \]
\[ y(t) = \sin t \]

for \( 0 \leq t \leq \pi \).
Solution

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Therefore the mass of the wire is

\[
m = \int_C \rho(x, y) \, ds
= \int_0^\pi k(1 - \sin t) \sqrt{(-\sin t)^2 + (\cos t)^2} \, dt
= k \int_0^\pi (1 - \sin t) \, dt
= k(t + \cos t) \bigg|_0^\pi
= k(\pi - 2)\]

Arc Length

Theorem

For any piecewise-smooth curve $C$,

$$
\int_C 1 \, ds
$$

gives the arc length of the curve $C$. 

Example

Find the arc length of the helix \((\sin t, t, \cos t)\) for \(0 \leq t \leq \pi\).
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\[
\begin{align*}
  s &= \int_C 1 \, ds \\
  &= \int_0^\pi \sqrt{(\cos t)^2 + (1)^2 + (-\sin t)^2} \, dt \\
  &= \int_0^\pi \sqrt{2} \, dt \\
  &= \pi \sqrt{2}
\end{align*}
\]
Smoothness

Definition
We say that curve $C$ is smooth if $C$ is described parametrically by $(x(t), y(t), z(t))$ for $a \leq t \leq b$, where $x(t)$, $y(t)$, and $z(t)$ have continuous first derivatives and

$$[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2 \neq 0$$

for $a \leq t \leq b$. 

Definition
Suppose that curve $C$ is $C = C_1 \cup C_2 \cup \ldots \cup C_n$ where each of the $C_1$, $C_2$, ..., $C_n$ is smooth and the terminal point of $C_i$ is the initial point of $C_{i+1}$ for $i = 1, 2, \ldots, n-1$. In this case we say that $C$ is piecewise-smooth.
Smoothness

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We say that curve $C$ is **smooth** if $C$ is described parametrically by $(x(t), y(t), z(t))$ for $a \leq t \leq b$, where $x(t)$, $y(t)$, and $z(t)$ have continuous first derivatives and

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Result

Theorem
Suppose that \( f(x, y, z) \) is a continuous function in some region \( D \) containing the oriented curve \( C \). Then if \( C \) is piecewise-smooth with \( C = C_1 \cup C_2 \cup \cdots \cup C_n \), where \( C_1, C_2, \ldots, C_n \) are smooth and the terminal point of \( C_i \) is the initial point of \( C_{i+1} \) for \( i = 1, 2, \ldots, n - 1 \), we have

\[
\int_{-C} f(x, y, z) \, ds = \int_C f(x, y, z) \, ds
\]

and

\[
\int_C f(x, y, z) \, ds = \int_{C_1} f(x, y, z) \, ds + \int_{C_2} f(x, y, z) \, ds + \cdots + \int_{C_n} f(x, y, z) \, ds.
\]
Evaluate the line integral

\[ \int_C (x + y) \, ds \]

where $C$ is the right-angled path from $(1, 0)$ to $(1, 1)$ to $(0, 1)$. 
Example (2 of 3)

\[ C_1 : \begin{cases} x = 1 \\ y = t \end{cases} \text{ for } 0 \leq t \leq 1 \]

\[ C_2 : \begin{cases} x = 1 - t \\ y = 1 \end{cases} \text{ for } 0 \leq t \leq 1 \]
\[ \int_C (x + y) \, ds = \int_{C_1} (x + y) \, ds + \int_{C_2} (x + y) \, ds \]
\[ = \int_0^1 (1 + t) \, dt + \int_0^1 (1 - t + 1) \, dt \]
\[ = \frac{3}{2} + \int_0^1 (2 - t) \, dt \]
\[ = \frac{3}{2} + \frac{3}{2} = 3 \]
Definition

The **line integral of** \( f(x, y, z) \) **with respect to** \( x \) along the oriented curve \( C \) in three-dimensional space is

\[
\int_C f(x, y, z) \, dx = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i) \Delta x_i
\]

provided the limit exists and is the same for every choice of evaluation points.
Definition
The **line integral of** $f(x, y, z)$ **with respect to** $y$ **along the oriented curve** $C$ **in three-dimensional space** is

\[
\int_C f(x, y, z) \, dy = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i) \Delta y_i
\]

provided the limit exists and is the same for every choice of evaluation points.
Definition

The **line integral of** $f(x, y, z)$ **with respect to** $z$ along the oriented curve $C$ in three-dimensional space is

$$\int_C f(x, y, z) \, dz = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i) \Delta z_i$$

provided the limit exists and is the same for every choice of evaluation points.
Theorem (Evaluation Theorem)

Suppose that $f(x, y, z)$ is continuous in a region $D$ containing the curve $C$ and that $C$ is described parametrically by $(x(t), y(t), z(t))$ for $a \leq t \leq b$, where $x(t)$, $y(t)$, and $z(t)$ have continuous first derivatives. Then

\[
\int_C f(x, y, z) \, dx = \int_a^b f(x(t), y(t), z(t))x'(t) \, dt,
\]

\[
\int_C f(x, y, z) \, dy = \int_a^b f(x(t), y(t), z(t))y'(t) \, dt,
\]

\[
\int_C f(x, y, z) \, dz = \int_a^b f(x(t), y(t), z(t))z'(t) \, dt.
\]
Examples

Compute the following line integrals over the path $C$ which is the portion of the parabola $x = y^2$ from $(1, 1)$ to $(1, -1)$.

- $\int_C (x + y) \, dx$
- $\int_C (x + y) \, dy$
- $\int_C (x + y) \, ds$
Illustration of Curve $C$
Solution

The curve $C$ is parameterized for $-1 \leq t \leq 1$ as

$$x = t^2 \quad y = -t$$

and then

$$\int_C (x + y) \, dx = \int_{-1}^{1} (t^2 - t)2t \, dt = -\frac{4}{3}$$

$$\int_C (x + y) \, dy = \int_{-1}^{1} (t^2 - t)(-1) \, dt = -\frac{2}{3}$$

$$\int_C (x + y) \, ds = \int_{-1}^{1} (t^2 - t)\sqrt{(2t)^2 + (-1)^2} \, dt \approx 1.21267$$
Theorem

Suppose that $f(x, y, z)$ is a continuous function in some region $D$ containing the oriented curve $C$. Then, the following hold.

1. If $C$ is piecewise-smooth, then

$$\int_{-C} f(x, y, z) \, dx = -\int_C f(x, y, z) \, dx.$$

2. If $C = C_1 \cup C_2 \cup \cdots \cup C_n$, where $C_1, C_2, \ldots, C_n$ are smooth and the terminal point of $C_i$ is the initial point of $C_{i+1}$ for $i = 1, 2, \ldots, n - 1$, then

$$\int_C f(x, y, z) \, dx = \sum_{i=1}^n \int_{C_i} f(x, y, z) \, dx.$$

Remark: similar results hold for integrals with respect to $y$ and $z$. 
Example

Calculate the line integral

$$\int_C 2x \ dy + 4y \ dx$$

where $C$ consists of the line segment from $(0, 0)$ to $(1, 0)$ followed by the quarter circle to $(0, 1)$ followed by the line segment to $(0, 0)$. 
Parameterization of $C$

$C_1 : \begin{cases} x = t \\ y = 0 \end{cases}$ for $0 \leq t \leq 1$

$C_2 : \begin{cases} x = \cos t \\ y = \sin t \end{cases}$ for $0 \leq t \leq \frac{\pi}{2}$

$C_3 : \begin{cases} x = 0 \\ y = 1 - t \end{cases}$ for $0 \leq t \leq 1$
Solution

\[ \int_C 2x \, dy + 4y \, dx = \int_{C_1} 2x \, dy + 4y \, dx + \int_{C_2} 2x \, dy + 4y \, dx + \int_{C_3} 2x \, dy + 4y \, dx \]

\[ = \int_{C_1} 2x \, dy + 4y \, dx + \int_{C_2} 2x \, dy + 4y \, dx \]

\[ + \int_{C_3} 2x \, dy + 4y \, dx \]

\[ = 0 + 0 \]

\[ = \int_{C_2} 2x \, dy + 4y \, dx \]

\[ = \int_0^{\pi/2} (2 \cos^2 t - 4 \sin^2 t) \, dt \]

\[ = -\frac{\pi}{2} \]
**Work**

Suppose \( \mathbf{F}(x, y, z) = \langle f_1(x, y, z), f_2(x, y, z), f_3(x, y, z) \rangle \) represents a vector field function representing the force present on an object at location \((x, y, z)\).

The work done in moving the object along curve \( C \) parametrized by \((x(t), y(t), z(t))\) where \( a \leq t \leq b \) is

\[
W = \int_a^b \mathbf{F}(x, y, z) \cdot \mathbf{r}'(t) \, dt
\]

\[
= \int_C f_1(x, y, z) \, dx + \int_C f_2(x, y, z) \, dy + \int_C f_3(x, y, z) \, dz
\]

\[
= \int_C \langle f_1(x, y, z), f_2(x, y, z), f_3(x, y, z) \rangle \cdot \langle dx, dy, dz \rangle
\]

\[
= \int_C \mathbf{F}(x, y, z) \cdot d\mathbf{r}
\]
Example

Compute the work done by the force field \( \mathbf{F}(x, y, z) = \langle xy, 3z, 1 \rangle \) along the quarter ellipse parametrized by

\[
\begin{align*}
  x &= 2 \cos t \\
  y &= 3 \sin t \\
  z &= 1
\end{align*}
\]

from \((2, 0, 1)\) to \((0, 3, 1)\).
Illustration of Vector Field and Curve
Calculation of Work

\[ W = \int_C \mathbf{F}(x, y, z) \cdot d\mathbf{r} \]

\[ = \int_0^{\pi/2} \langle 6 \cos t \sin t, 3, 1 \rangle \cdot \langle -2 \sin t, 3 \cos t, 0 \rangle \, dt \]

\[ = \int_0^{\pi/2} 9 \cos t - 12 \cos t \sin^2 t \, dt \]

\[ = \int_0^{\pi/2} 9 \cos t \, dt - 12 \int_0^{\pi/2} \cos t \sin^2 t \, dt \]

\[ = 9 - 12 \int_0^1 u^2 \, du \]

\[ = 5 \]
Remarks

If the orientation of the curve is generally in the same direction as the vector field, the force adds energy to the object and thus does *positive* work.

If the orientation of the curve is generally in the direction opposite the vector field, the force opposes the motion of the object and thus does *negative* work.
Homework

- Read Section 14.2.
- Exercises: 1–41 odd.