Lines and Planes in Space
MATH 311, *Calculus III*

J. Robert Buchanan

Department of Mathematics

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Lines and Points
Lines and Vectors

We can describe lines in \( \mathbb{R}^3 \) by referring to vectors in \( V_3 \).

Consider a nonzero vector \( \mathbf{a} \) and a point \( P_0 = (x_0, y_0, z_0) \). The vector with initial point \( P_0 \) in the direction of \( \mathbf{a} = \langle a_1, a_2, a_3 \rangle \) is

\[
\overrightarrow{P_0P} = t \mathbf{a}
\]

where \( t \) is a scalar.

If point \( P = (x, y, z) \) then

\[
\langle x - x_0, y - y_0, z - z_0 \rangle = t\langle a_1, a_2, a_3 \rangle.
\]

This vector equation can be re-written as a system of 3 linear equations.
The **parametric equations** for the line through \((x_0, y_0, z_0)\) in the direction of vector \(\mathbf{a} = \langle a_1, a_2, a_3 \rangle\):

\[
\begin{align*}
x &= x_0 + t a_1 \\
y &= y_0 + t a_2 \\
z &= z_0 + t a_3
\end{align*}
\]
The parametric equations for the line through \((x_0, y_0, z_0)\) in the direction of vector \(\mathbf{a} = \langle a_1, a_2, a_3 \rangle\):

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\begin{align*}
x &= x_0 + t a_1 \\
y &= y_0 + t a_2 \\
z &= z_0 + t a_3
\end{align*}
\]

If we combine the three equations and eliminate the parameter \(t\), then the symmetric equations for the line are

\[
\frac{x - x_0}{a_1} = \frac{y - y_0}{a_2} = \frac{z - z_0}{a_3}.
\]
Example (1 of 3)

Let $P_0 = (5, 1, 3)$ and $\mathbf{a} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ and find the equation of the line through $P_0$ in the direction of $\mathbf{a}$ in parametric and symmetric form.
Example (1 of 3)

Let $P_0 = (5, 1, 3)$ and $\mathbf{a} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ and find the equation of the line through $P_0$ in the direction of $\mathbf{a}$ in parametric and symmetric form.

**Parametric Form:**

\[
\begin{align*}
    x &= 5 + t \\
    y &= 1 + 4t \\
    z &= 3 - 2t
\end{align*}
\]

**Symmetric Form:**

\[
\frac{x - 5}{1} = \frac{y - 1}{4} = \frac{z - 3}{-2}
\]
Example (2 of 3)

Find the equation of the line which passes through the points $A = (2, 4, -3)$ and $B = (3, -1, 1)$. Where does this line intersect the $xy$-plane?
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Find the equation of the line which passes through the points $A = (2, 4, -3)$ and $B = (3, -1, 1)$. Where does this line intersect the $xy$-plane?

The line is parallel to the vector:

$$
v = \overrightarrow{AB} = \langle 3 - 2, -1 - 4, 1 - (-3) \rangle = \langle 1, -5, 4 \rangle.
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The parametric form of the line is:

$$\begin{align*}
x &= 2 + t \\
y &= 4 - 5t \\
z &= -3 + 4t
\end{align*}$$

The point of intersection is $(x, y, z) = (11/4, 1/4, 0)$. 
Find the equation of the line which passes through the points $A = (2, 4, -3)$ and $B = (3, -1, 1)$. Where does this line intersect the $xy$-plane?

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The parametric form of the line is:

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\begin{align*}
x &= 2 + t \\
y &= 4 - 5t \\
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\end{align*}
$$

The line intersects the $xy$-plane when $z = 0$ which implies $t = 3/4$.

The point of intersection is $(x, y, z) = (11/4, 1/4, 0)$. 

Example (3 of 3)

Do the following two lines intersect?

\[
\begin{align*}
  x &= 1 + t \\
  y &= -2 + 3t \\
  z &= 4 - t \\
  x &= 2s \\
  y &= 3 + s \\
  z &= -3 + 4s
\end{align*}
\]

If the lines intersect the lines must have a point in common. Solving the system of two equations in two unknowns:

\[
\begin{align*}
  1 + t &= 2s \\
  -2 + 3t &= 3 + s \\
  4 - t &= -3 + 4s
\end{align*}
\]

implies \( s = \frac{8}{5} \) and \( t = \frac{11}{5} \). However, using these \( s \) and \( t \) values makes the \( z \)-coordinates unequal. Thus the lines do not intersect.
Example (3 of 3)

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\begin{align*}
x & = 1 + t \\
y & = -2 + 3t \\
z & = 4 - t
\end{align*}
\quad
\begin{align*}
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y & = 3 + s \\
z & = -3 + 4s
\end{align*}
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\[ y = -2 + 3t \quad y = 3 + s \]
\[ z = 4 - t \quad z = -3 + 4s \]

If the lines intersect the lines must have a point in common. Solving the system of two equations in two unknowns:

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However, using these \( s \) and \( t \) values makes the \( z \)-coordinates unequal. Thus the lines do not intersect.
Parallel and Orthogonal Lines

Definition
Let $l_1$ and $l_2$ be two lines in $\mathbb{R}^3$, with parallel vectors $\mathbf{a}$ and $\mathbf{b}$, respectively, and let $\theta$ be the angle between $\mathbf{a}$ and $\mathbf{b}$.

1. The lines $l_1$ and $l_2$ are **parallel** whenever $\mathbf{a}$ and $\mathbf{b}$ are parallel.
2. If $l_1$ and $l_2$ intersect, then
   - the angle between $l_1$ and $l_2$ is $\theta$ and
   - the lines $l_1$ and $l_2$ are **orthogonal** whenever $\mathbf{a}$ and $\mathbf{b}$ are orthogonal.

Definition
Nonparallel, non-intersecting lines are called **skew** lines.
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Planes in $\mathbb{R}^3$

A plane can be thought of as the collection of all lines orthogonal to a given line.
Planes and Vectors

If point $P_0 = (x_0, y_0, z_0)$ lies in the plane and vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is normal to the plane (i.e., orthogonal to every line in the plane) and if point $P = (x, y, z)$ is an arbitrary point in the plane, then

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- $\overrightarrow{P_0P}$ is orthogonal to $\mathbf{a}$,
- $\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a_1, a_2, a_3 \rangle = 0$, and
Planes and Vectors

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$\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a_1, a_2, a_3 \rangle = 0$, and

the equation of the plane is

$$a_1(x - x_0) + a_2(y - y_0) + a_3(z - z_0) = 0.$$
Example (1 of 2)

Find the equation of the plane through the point \((2, 4, -1)\) with normal vector \(\mathbf{n} = \langle 2, 3, 4 \rangle\).
Find the equation of the plane through the point $(2, 4, -1)$ with normal vector $\mathbf{n} = \langle 2, 3, 4 \rangle$.

$$\langle x - 2, y - 4, z - (-1) \rangle \cdot \langle 2, 3, 4 \rangle = 0$$

$$2x + 3y + 4z = 12$$
Example (2 of 2)

Find the equation of the plane containing the points $P = (1, 3, 2), Q = (3, -1, 6), \text{ and } R = (5, 2, 0)$. 

First we must find a vector orthogonal to the plane containing the three points. Let $a = \overrightarrow{PQ} = \langle 2, -4, 4 \rangle$ and let $b = \overrightarrow{PR} = \langle 4, -1, -2 \rangle$, then using the cross product we have a vector perpendicular to the plane.

$$n = a \times b = \langle 12, 20, 14 \rangle$$

The equation of the plane is

$$\langle x - 1, y - 3, z - 2 \rangle \cdot \langle 12, 20, 14 \rangle = 0$$

or

$$6x + 10y + 7z = 50$$
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Remarks

The equation of a plane in $\mathbb{R}^3$ has the form:

$$ax + by + cz = d$$

where not all of $a$, $b$, and $c$ can be zero.
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- If $ax + by + cz = d$ defines a plane, then $\mathbf{v} = \langle a, b, c \rangle$ is normal to the plane.
Remarks

- The equation of a plane in $\mathbb{R}^3$ has the form:

  \[ ax + by + cz = d \]

  where not all of $a$, $b$, and $c$ can be zero.

- If $ax + by + cz = d$ defines a plane, then $\mathbf{v} = \langle a, b, c \rangle$ is normal to the plane.

- An easy method for sketching a plane is to sketch the simplex of the plane defined by its intersections with the coordinate axes.
Parallel and Orthogonal Planes

Definition
Two planes with normal vectors \( \mathbf{a} \) and \( \mathbf{b} \) are

1. **parallel** if \( \mathbf{a} \) and \( \mathbf{b} \) are parallel.
2. **orthogonal** if \( \mathbf{a} \) and \( \mathbf{b} \) are orthogonal.
Example (1 of 3)

Are the planes defined by $x + 2y - 3z = 4$ and $2x + 4y - 6z = 1$ parallel?
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Are the planes defined by $x + 2y - 3z = 4$ and $2x + 4y - 6z = 1$ parallel?
A normal vector to the first plane is $\mathbf{a} = \langle 1, 2, -3 \rangle$ while a normal vector to the second plane is $\mathbf{b} = \langle 2, 4, -6 \rangle$. 
Are the planes defined by \( x + 2y - 3z = 4 \) and \( 2x + 4y - 6z = 1 \) parallel?

A normal vector to the first plane is \( \mathbf{a} = \langle 1, 2, -3 \rangle \) while a normal vector to the second plane is \( \mathbf{b} = \langle 2, 4, -6 \rangle \).

Since \( \mathbf{b} \) is a scalar multiple of \( \mathbf{a} \) (namely \( \mathbf{b} = 2\mathbf{a} \)) then the normal vectors are parallel, which implies the original planes are parallel.
Find the angle between the planes

\[ x + y + z = 1 \]
\[ x - 2y + 3z = 2. \]
Example (2 of 3)

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The angle between the planes will be the angle between their normal vectors.
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Let \( \mathbf{a} = \langle 1, 1, 1 \rangle \) and \( \mathbf{b} = \langle 1, -2, 3 \rangle \), then

\[
\mathbf{a} \cdot \mathbf{b} = \| \mathbf{a} \| \| \mathbf{b} \| \cos \theta
\]

\[
2 = \sqrt{42} \cos \theta
\]

\[
\theta \approx 1.25707 \approx 72.02^\circ
\]
Example (3 of 3)

Find the line of intersection of the two planes

\[ x + y + z = 1 \]
\[ x - 2y + 3z = 2. \]
Example (3 of 3)

Find the line of intersection of the two planes

\[
\begin{align*}
    x + y + z &= 1 \\
    x - 2y + 3z &= 2.
\end{align*}
\]

Eliminate \( x \) from the two equations and then treat \( z \) as the parameter.

\[
\begin{align*}
    1 - y - z &= x = 2 + 2y - 3z \\
    y &= -\frac{1}{3} + \frac{2}{3}z
\end{align*}
\]
Example (3 of 3)

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\[ x + y + z = 1 \]
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\[ 1 - y - z = x = 2 + 2y - 3z \]
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Parametric Form:

\[ x = \frac{4}{3} - \frac{5}{3}t, \quad y = -\frac{1}{3} + \frac{2}{3}t, \quad z = t \]
The distance from $P_2$ to the plane is $\left| \text{comp}_a \overrightarrow{P_1P_2} \right|$. 
Distance from a Point to a Plane

If \( \mathbf{a} = \langle a, b, c \rangle \), \( P_1 = (x_1, y_1, z_1) \), and \( P_2 = (x_2, y_2, z_2) \) then

\[
\text{comp}_a \overrightarrow{P_1 P_2} = \frac{\mathbf{a} \cdot \overrightarrow{P_1 P_2}}{\| \mathbf{a} \|}
\]

\[
= \frac{\langle a, b, c \rangle \cdot \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle}{\sqrt{a^2 + b^2 + c^2}}
\]

\[
= \frac{ax_2 + by_2 + cz_2 - (ax_1 + by_1 + cz_1)}{\sqrt{a^2 + b^2 + c^2}}
\]

distance \( = \frac{|ax_2 + by_2 + cz_2 + d|}{\sqrt{a^2 + b^2 + c^2}} \)
Example (1 of 3)

Find the distance from \( (\frac{1}{2}, 0, 1) \) to \( 5x + y - z = 1 \).
Example (1 of 3)

Find the distance from \( (\frac{1}{2}, 0, 1) \) to \( 5x + y - z = 1 \).
Rather than trying to apply the distance formula from memory it may be easier to recall that the distance from the point to the plane is \( |\text{comp}_a \overrightarrow{P_1P_2}| \), where \( a \) is the normal vector to the plane and we are free to pick point \( P_1 \) to be any point in the plane.
Example (1 of 3)

Find the distance from \((\frac{1}{2}, 0, 1)\) to \(5x + y - z = 1\). Rather than trying to apply the distance formula from memory it may be easier to recall that the distance from the point to the plane is \(|\text{comp}_a \overrightarrow{P_1 P_2}|\), where \(a\) is the normal vector to the plane and we are free to pick point \(P_1\) to be any point in the plane. Let \(P_2 = (\frac{1}{2}, 0, 1)\), let \(P_1 = (0, 1, 0)\), and \(a = \langle 5, 1, -1 \rangle\), then

\[
|\text{comp}_a \overrightarrow{P_1 P_2}| = \left| \frac{\langle 1/2, -1, 1 \rangle \cdot \langle 5, 1, -1 \rangle}{\|\langle 5, 1, -1 \rangle\|} \right| = \frac{1}{6\sqrt{3}}.
\]
Example (2 of 3)

Find the distance between the planes

\[ 10x + 2y - 2z = 5 \]
\[ 5x + y - z = 1 \]
Example (2 of 3)

Find the distance between the planes

\[ 10x + 2y - 2z = 5 \]
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If the planes were not parallel we could immediately declare the distance between them is 0.
Example (2 of 3)

Find the distance between the planes

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\[ 5x + y - z = 1 \]

If the planes were not parallel we could immediately declare the distance between them is 0.

Pick a point in the second plane, say \( P_2 = (0, 1, 0) \) and find its distance to the first plane.
Example (2 of 3)

Find the distance between the planes

\[ 10x + 2y - 2z = 5 \]
\[ 5x + y - z = 1 \]

If the planes were not parallel we could immediately declare the distance between them is 0.

Pick a point in the second plane, say \( P_2 = (0, 1, 0) \) and find its distance to the first plane.

Note that \( P_1 = (1/2, 0, 0) \) is in the first plane and the normal vector to the first plane is \( \mathbf{a} = \langle 10, 2, -2 \rangle \).

\[
|\text{comp}_a \overrightarrow{P_1 P_2}| = \left| \frac{\langle -1/2, 1, 0 \rangle \cdot \langle 10, 2, -2 \rangle}{\|\langle 10, 2, -2 \rangle\|} \right| = \frac{1}{2\sqrt{3}}
\]
Example (3 of 3)
Find the distance between the skew lines

\[ x = 1 + t, \quad y = -2 + 3t, \quad z = 4 - t, \quad \text{and} \]
\[ x = 2s, \quad y = 3 + s, \quad z = -3 + 4s. \]
Example (3 of 3)

Find the distance between the skew lines

\[ x = 1 + t, \quad y = -2 + 3t, \quad z = 4 - t, \quad \text{and} \]
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We must find the equations of two parallel planes, each containing one of the lines above.
Example (3 of 3)

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\[ x = 2s, \quad y = 3 + s, \quad z = -3 + 4s. \]

We must find the equations of two parallel planes, each containing one of the lines above.

The common normal vector shared by the planes must be perpendicular to both lines.

\[ \mathbf{a} = \langle 1, 3, -1 \rangle \times \langle 2, 1, 4 \rangle = \langle 13, -6, -5 \rangle \]
Example (3 of 3)

Find the distance between the skew lines

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The common normal vector shared by the planes must be perpendicular to both lines.

\[ \mathbf{a} = \langle 1, 3, -1 \rangle \times \langle 2, 1, 4 \rangle = \langle 13, -6, -5 \rangle \]

Let \( P_1 = (1, -2, 4) \) be a point on the first line and let \( P_2 = (0, 3, -3) \) be a point on the second line.

\[ |\text{comp}_a \overrightarrow{P_1P_2}| = \left| \frac{\langle -1, 5, -7 \rangle \cdot \langle 13, -6, -5 \rangle}{\|\langle 13, -6, -5 \rangle\|} \right| = \sqrt{\frac{32}{115}} \]
Homework

- Read Section 10.5.
- Exercises: 1–61 odd.