Motion in Space
MATH 311, *Calculus III*

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Suppose the position vector of a moving object is given by

\[ \mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle, \]

and the magnitude of the tangent vector is

\[ \| \mathbf{r}'(t) \| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}. \]
Suppose the position vector of a moving object is given by

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Recall the arc length of a parametric curve in three dimensions is

\[ s(t) = \int_{t_0}^{t} \sqrt{[f'(u)]^2 + [g'(u)]^2 + [h'(u)]^2} \, du = \int_{t_0}^{t} \|r'(u)\| \, du. \]
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\[ s'(t) = \| r'(t) \| \]

so \( \| r'(t) \| \) is the **speed** of the moving object.
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so \( \|r'(t)\| \) is the **speed** of the moving object.

Consequently, \( r'(t) \) is the **velocity vector** and \( r''(t) \) is the **acceleration vector**.
Find the velocity and acceleration vectors of an object whose position vector is

\[ r(t) = \left\langle \frac{2}{3} t^3 + t + 1, t^3 - t, e^t \right\rangle. \]
Example (1 of 2)

Find the velocity and acceleration vectors of an object whose position vector is

\[ r(t) = \left\langle \frac{2}{3} t^3 + t + 1, t^3 - t, e^t \right\rangle. \]

\[ v(t) = r'(t) = \left\langle 2t^2 + 1, 3t^2 - 1, e^t \right\rangle \]

\[ a(t) = r''(t) = \left\langle 4t, 6t, e^t \right\rangle \]
Example (2 of 2)

The acceleration vector of a moving object is

\[
a(t) = \left\langle \frac{t}{6}, \sin t, 0 \right\rangle,
\]

while its initial velocity is \( v(0) = \langle 2, 0, 3 \rangle \) and its initial position is \( r(0) = \langle 0, 0, 1 \rangle \). Find the velocity and position vectors as a function of \( t \) for this object.
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\[
\mathbf{v}(t) - \mathbf{v}(0) = \int_0^t \left\langle \frac{s}{6}, \sin s, 0 \right\rangle \, ds
\]

\[
\mathbf{v}(t) = \langle 2, 0, 3 \rangle + \left\langle \frac{t^2}{12}, 1 - \cos t, 0 \right\rangle
\]

\[
= \left\langle 2 + \frac{t^2}{12}, 1 - \cos t, 3 \right\rangle
\]
Finding the Position Vector

\[ r(t) - r(0) = \int_0^t \left\langle 2 + \frac{s^2}{12}, 1 - \cos s, 3 \right\rangle \, ds \]

\[ r(t) = \langle 0, 0, 1 \rangle + \left\langle 2t + \frac{t^3}{36}, t - \sin t, 3t \right\rangle \]

\[ = \left\langle 2t + \frac{t^3}{36}, t - \sin t, 1 + 3t \right\rangle \]
Newton’s Second Law

One of the most fundamental physical laws is Newton’s second law of motion which states that the force vector acting on an object is the product of the object’s mass (a scalar) and the object’s acceleration vector. This is stated concisely as $F = ma$. 
Example

Find the force acting on an object moving along an elliptical path

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.
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Let the position vector of the object be

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\mathbf{r}(t) = \langle a \cos \omega t, b \sin \omega t \rangle
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then
\[
F(t) = ma(t) = m r''(t) = -m \omega^2 \langle a \cos t, b \sin t \rangle = -m \omega^2 r(t).
\]
Centripetal Motion

\[ \mathbf{r}(t) = \langle a \cos \omega t, b \sin \omega t \rangle \]
Suppose an object is launched at an angle $\theta$ with respect to the horizontal from a height $h$. The object is given an initial speed of $s_0$. Find the position vector describing the path the object takes.
Equations of Motion

Suppose an object is launched at an angle $\theta$ with respect to the horizontal from a height $h$. The object is given an initial speed of $s_0$. Find the position vector describing the path the object takes. Treating this as motion in the $xy$-plane, we have an acceleration vector $\mathbf{a}(t) = \langle 0, -g \rangle$ and initial condition vectors

$$\mathbf{v}_0 = s_0 \langle \cos \theta, \sin \theta \rangle \quad \text{and} \quad \mathbf{r}_0 = \langle 0, h \rangle.$$
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$$
\mathbf{v}_0 = s_0 \langle \cos \theta, \sin \theta \rangle \quad \text{and} \quad \mathbf{r}_0 = \langle 0, h \rangle.
$$

Thus the velocity and position vectors are

$$
\mathbf{v}(t) = \langle s_0 \cos \theta, s_0 \sin \theta - gt \rangle
$$

$$
\mathbf{r}(t) = \langle (s_0 \cos \theta)t, h + (s_0 \sin \theta)t - \frac{1}{2}gt^2 \rangle.
$$
Rotational Motion

We can adapt Newton’s second law for spinning objects.

\( \tau \)  torque (scalar, \( \tau = ||\tau|| \))

\( I \)  moment of inertia, measure of the force required to start an object rotating

\( \theta \)  angle of displacement

\( \omega \)  angular velocity

\( \alpha \)  angular acceleration

\[ \alpha(t) = \omega'(t) = \theta''(t) \]
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**\( \theta \)** angle of displacement

**\( \omega \)** angular velocity

**\( \alpha \)** angular acceleration

\[
\alpha(t) = \omega'(t) = \theta''(t)
\]

\[
\tau = I\alpha
\]
Example

A merry-go-round of radius 6 feet and moment of inertia $I = 12$ rotates at 5 radians per second. Find the constant force applied tangent to the edge of the merry-go-round needed to stop the merry-go-round in 3 seconds.
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$$5 = \omega(3) - \omega(0) = \int_0^3 \alpha \, dt = 3\alpha$$

implies $\alpha = 5/3$.

$$\tau = (12)(5/3) = 20 = \| \mathbf{r} \times \mathbf{F} \| = \| \mathbf{r} \| \| \mathbf{F} \| = (6)\| \mathbf{F} \|$$

which implies $\| \mathbf{F} \| = 10/3$ pounds.
Momentum

Suppose an object has mass \( m \) and velocity \( \mathbf{v} \).

- The object’s **linear momentum** is \( \mathbf{p}(t) = m \mathbf{v}(t) \).
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- The object’s **linear momentum** is \( \mathbf{p}(t) = m\mathbf{v}(t) \).
- The object’s **angular momentum** is \( \mathbf{L}(t) = \mathbf{r}(t) \times m\mathbf{v}(t) \).

**Example**

Show that torque is the derivative of angular momentum.
Momentum

Suppose an object has mass $m$ and velocity $\mathbf{v}$.

- The object’s **linear momentum** is $\mathbf{p}(t) = m \mathbf{v}(t)$.
- The object’s **angular momentum** is $\mathbf{L}(t) = \mathbf{r}(t) \times m \mathbf{v}(t)$.

**Example**

Show that torque is the derivative of angular momentum.

\[
\mathbf{L}'(t) = \mathbf{r}'(t) \times m \mathbf{v}(t) + \mathbf{r}(t) \times m \mathbf{v}'(t) \\
= \mathbf{v}(t) \times m \mathbf{v}(t) + \mathbf{r}(t) \times m \mathbf{a}(t) \\
= \mathbf{0} + \mathbf{r}(t) \times \mathbf{F}(t) \\
= \tau(t)
\]
A projectile of mass 10 kg is launched to the east from a height of 1 m at a speed of 10 m/s. The launch angle is 45°. The projectile spins as it flies and thus is subject to a Magnus force of magnitude 2 N in the southerly direction. Find the position of the projectile, its landing location, and its speed at impact.
Assumptions:

- East is the positive $x$-direction and south is the negative $y$-direction.
- The only forces acting on the projectile are gravity and the Magnus force.
- The projectile is launched one meter above the origin on the positive $z$-axis.
Example (1 of 3)

Assumptions:

▶ East is the positive $x$-direction and south is the negative $y$-direction.
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▶ The projectile is launched one meter above the origin on the positive $z$-axis.

\[
\mathbf{m} \mathbf{a} = \mathbf{F} \\
= 10 \langle 0, 0, -9.8 \rangle + \langle 0, -2, 0 \rangle \\
\mathbf{a} = \langle 0, -\frac{1}{5}, -9.8 \rangle
\]
Example (2 of 3)

\[ r''(t) = \langle 0, -\frac{1}{5}, -9.8 \rangle \]
\[ r'(t) = \langle 0, -\frac{t}{5}, -9.8t \rangle + v(0) \]

where \( v(0) = 10\langle \cos 45^\circ, 0, \sin 45^\circ \rangle = \langle 5\sqrt{2}, 0, 5\sqrt{2} \rangle \).
Example (2 of 3)

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\[ r'(t) = \langle 5\sqrt{2}, -\frac{t}{5}, 5\sqrt{2} - 9.8t \rangle \]

\[ r(t) = \langle 5\sqrt{2}t, -\frac{t^2}{10}, 5\sqrt{2}t - 4.9t^2 \rangle + r(0) \]

where \( r(0) = \langle 0, 0, 1 \rangle. \)
Example (2 of 3)

\[
\begin{align*}
\mathbf{r}''(t) &= \langle 0, -\frac{1}{5}, -9.8 \rangle \\
\mathbf{r}'(t) &= \langle 0, -\frac{t}{5}, -9.8t \rangle + \mathbf{v}(0)
\end{align*}
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where \( \mathbf{v}(0) = 10\langle \cos 45^\circ, 0, \sin 45^\circ \rangle = \langle 5\sqrt{2}, 0, 5\sqrt{2} \rangle \).

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\mathbf{r}(t) &= \langle 5\sqrt{2}t, -\frac{t^2}{10}, 5\sqrt{2}t - 4.9t^2 \rangle + \mathbf{r}(0)
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where \( \mathbf{r}(0) = \langle 0, 0, 1 \rangle \).

\[
\mathbf{r}(t) = \langle 5\sqrt{2}t, -\frac{t^2}{10}, 1 + 5\sqrt{2}t - 4.9t^2 \rangle
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Example (3 of 3)

Since

\[ \mathbf{r}(t) = \langle 5\sqrt{2}t, -\frac{t^2}{10}, 1 + 5\sqrt{2}t - 4.9t^2 \rangle \]

the projectile lands when

\[ 1 + 5\sqrt{2}t - 4.9t^2 = 0 \implies t \approx 1.57283, \]

which implies

\[ \mathbf{r}(1.57283) \approx \langle 11.1216, -0.247379, 0 \rangle \]
\[ \|\mathbf{r}'(1.57283)\| \approx 10.9407 \]
Homework

- Read Section 11.3.
- Exercises: 1–55 odd.