Stokes’ Theorem
MATH 311, *Calculus III*

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Background (1 of 2)

**Recall:** Green’s Theorem,

\[
\oint_C M(x, y) \, dx + N(x, y) \, dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA
\]

where \( C \) is a piecewise smooth, positively oriented, simple closed curve in the \( xy \)-plane enclosing region \( R \).

- Define the vector field \( \mathbf{F}(x, y) = \langle M(x, y), N(x, y), 0 \rangle \).
- Note that \( \nabla \times \mathbf{F} = \langle 0, 0, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \rangle \).
- The component of \( \nabla \times \mathbf{F} \) along \( \mathbf{k} \) is \n  \[
  (\nabla \times \mathbf{F}) \cdot \mathbf{k} = \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k} \cdot \mathbf{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}
\]
Thus we have developed the vector form of Green’s Theorem

\[ \oint_C M(x, y) \, dx + N(x, y) \, dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA \]

\[ \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dA \]

Today we will generalize this result to three dimensions.
Using the right-hand rule, curve $C$ has positive orientation if it has the same orientation as the right hand’s fingers when the right thumb points in the direction of the normal $\mathbf{n}$ to surface $S$. Otherwise $C$ has negative orientation.
Stokes’ Theorem

Theorem (Stokes’ Theorem)
Suppose that $S$ is an oriented, piecewise-smooth surface with unit normal vector $\mathbf{n}$, bounded by the simple closed, piecewise-smooth boundary curve $\partial S$ having positive orientation. Let $\mathbf{F}(x, y, z)$ be a vector field whose components have continuous first partial derivatives in some open region containing $S$. Then,

$$
\int_{\partial S} \mathbf{F}(x, y, z) \cdot d\mathbf{r} = \int \int_{S} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS.
$$
Interpretation

Recalling that the unit tangent vector is

\[ T(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \quad \iff \quad \mathbf{r}'(t) = \|\mathbf{r}'(t)\| \mathbf{T}(t) \]

and that differential arc length is defined as

\[ ds = \|\mathbf{r}'(t)\| \, dt, \]

sometimes Stokes’ Theorem is written as

\[ \int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = \oint_{\partial S} \mathbf{F}(x, y, z) \cdot d\mathbf{r} = \oint_{\partial S} \mathbf{F} \cdot \mathbf{T} \, ds \]

where \( \mathbf{T} \) is the unit tangent in the direction of \( \partial S \).

**Interpretation:** The line integral of the tangential component of \( \mathbf{F} \) is equal to the flux of the curl of \( \mathbf{F} \). This integral is the average tendency of the flow of \( \mathbf{F} \) to rotate around path \( \partial S \).
Example (1 of 4)

Evaluate $\int_{\partial S} \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = -y^2\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$ and $\partial S$ is the intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. 
Example (2 of 4)

- Note that $\nabla \times \mathbf{F} = (1 + 2y)\mathbf{k}$.
- There are many surfaces which have $\partial S$ as a boundary, choose the elliptical disk bounded by $\partial S$ in the plane $y + z = 2$.
- The unit normal to this surface is $\mathbf{n} = \frac{1}{\sqrt{2}}\langle 0, 1, 1 \rangle$. 
Example (3 of 4)
Example (4 of 4)

According to Stokes’ Theorem

\[ \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_{S} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS \]

\[ = \iint_{S} (1 + 2y) \mathbf{k} \cdot \frac{1}{\sqrt{2}} \langle 0, 1, 1 \rangle \, dS \]

\[ = \iint_{S} \frac{1}{\sqrt{2}} (1 + 2y) \, dS \]

\[ = \iint_{R} (1 + 2y) \, dA \]

\[ = \int_{0}^{2\pi} \int_{0}^{1} (1 + 2r \sin \theta) r \, dr \, d\theta \]

\[ = \int_{0}^{2\pi} \int_{0}^{1} r \, dr \, d\theta \]

\[ = \pi \]
Example (1 of 4)

Evaluate \( \int \int_S ( \nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS \) where \( \mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} \)

and \( S \) is the part of the sphere \( x^2 + y^2 + z^2 = 4 \) that lies inside the cylinder \( x^2 + y^2 = 1 \) and above the \( xy \)-plane.
Surface $S$ is bounded by a circle formed by the intersection of the sphere of radius 2 and the cylinder of radius 1.

We can describe $\partial S$ using the vector-valued function

$$r(t) = \langle \cos t, \sin t, \sqrt{3} \rangle,$$

with $0 \leq t \leq 2\pi$. 
Example (3 of 4)
Example (4 of 4)

According to Stokes’ Theorem

\[ \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS \]

\[ = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} \]

\[ = \int_0^{2\pi} \mathbf{F}(\cos t, \sin t, \sqrt{3}) \cdot \langle -\sin t, \cos t, 0 \rangle \, dt \]

\[ = \sqrt{3} \int_0^{2\pi} (\cos^2 t - \sin^2 t) \, dt \]

\[ = \sqrt{3} \int_0^{2\pi} \cos 2t \, dt \]

\[ = 0 \]
An Identity

Show that \( \oint \mathbf{C} (f \nabla f) \cdot d\mathbf{r} = 0. \)

\[
\oint \mathbf{C} (f \nabla f) \cdot d\mathbf{r} = \iiint \mathbf{S} (\nabla \times (f \nabla f)) \cdot \mathbf{n} \, dS
\]

\[
= \iiint \mathbf{S} (\nabla \times \langle f \frac{\partial f}{\partial x}, f \frac{\partial f}{\partial y}, f \frac{\partial f}{\partial z} \rangle) \cdot \mathbf{n} \, dS
\]

\[
= \iiint \mathbf{S} 0 \cdot \mathbf{n} dS
\]

\[
= 0
\]
Let \( P = (x_0, y_0, z_0) \) be any point in a vector field \( \mathbf{F} \) and let \( S_a \) be a circular disk of radius \( a > 0 \) centered at \( P \). Let \( C_a \) be the boundary of \( S_a \).
Average value of \((\nabla \times \mathbf{F}) \cdot \mathbf{n}\) on surface \(S_a\):

\[
[(\nabla \times \mathbf{F}) \cdot \mathbf{n}]_{P_a} = \frac{1}{\pi a^2} \int \int_{S_a} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS
\]

where \(\pi a^2\) is the area of \(S_a\) and point \(P_a \in S_a\) by the Integral Mean Value Theorem.
Interpretation of the Curl (2 of 3)

Average value of \((\nabla \times \mathbf{F}) \cdot \mathbf{n}\) on surface \(S_a\):

\[
[(\nabla \times \mathbf{F}) \cdot \mathbf{n}]_{P_a} = \frac{1}{\pi a^2} \int \int_{S_a} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS
\]

where \(\pi a^2\) is the area of \(S_a\) and point \(P_a \in S_a\) by the Integral Mean Value Theorem.

By Stokes’ Theorem

\[
\oint_{C_a} \mathbf{F} \cdot d\mathbf{r} = \int \int_{S_a} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS
\]

\[
= \pi a^2 \left[(\nabla \times \mathbf{F}) \cdot \mathbf{n}\right]_{P_a}
\]
Interpretation of the Curl (3 of 3)

\[
[(\nabla \times \mathbf{F}) \cdot \mathbf{n}]_{Pa} = \frac{1}{\pi a^2} \oint_{C_a} \mathbf{F} \cdot d\mathbf{r}
\]

\[
\lim_{a \to 0^+} [(\nabla \times \mathbf{F}) \cdot \mathbf{n}]_{Pa} = \lim_{a \to 0^+} \frac{1}{\pi a^2} \oint_{C_a} \mathbf{F} \cdot d\mathbf{r}
\]

\[
[(\nabla \times \mathbf{F}) \cdot \mathbf{n}]_{P} = \lim_{a \to 0^+} \frac{1}{\pi a^2} \oint_{C_a} \mathbf{F} \cdot \mathbf{T} \, ds
\]

If \( \mathbf{F} \) describes the flow of a fluid then \( \oint_{C} \mathbf{F} \cdot \mathbf{T} \, ds \) is the \textbf{circulation around} \( C \), the average tendency of the fluid to circulate around the curve.
Remarks

- $[⟨∇ \times F⟩ \cdot n]_P$ attains its maximum when $∇ \times F$ is parallel to $n$.
- We can define $\text{rot } F = (∇ \times F) \cdot n$. This quantity is the rotation of the vector field at a point.
- $F$ is an irrotational vector field if and only if $∇ \times F = 0$. 
Example

Suppose $\mathbf{F} = \langle 3y, 4z, -6x \rangle$. Find the direction of the maximum value of $(\nabla \times \mathbf{F}) \cdot \mathbf{n}$. 
Example

Suppose $\mathbf{F} = \langle 3y, 4z, -6x \rangle$. Find the direction of the maximum value of $(\nabla \times \mathbf{F}) \cdot \mathbf{n}$.

Since $\nabla \times \mathbf{F} = \langle -4, 6, -3 \rangle$, the maximum of $(\nabla \times \mathbf{F}) \cdot \mathbf{n}$ will occur in the direction of

$$\mathbf{n} = \frac{\langle -4, 6, -3 \rangle}{\|\langle -4, 6, -3 \rangle\|} = \frac{1}{\sqrt{61}} \langle -4, 6, -3 \rangle.$$
Irrotational Vector Fields

Theorem
Suppose that $\mathbf{F}(x, y, z)$ has continuous partial derivatives throughout a simply connected region $D$, then $\nabla \times \mathbf{F} = \mathbf{0}$ in $D$ if and only if $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every simple closed curve $C$ in $D$. 
Proof (1 of 2)

- Suppose $\nabla \times \mathbf{F} = 0$.
- According to Stokes’ Theorem

\[
\oint \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = \iint_S 0 \, dS = 0.
\]
Proof (2 of 2)

> Suppose $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every simple closed curve $C$.
> If at some point $P$, $(\nabla \times \mathbf{F}) \neq \mathbf{0}$, then by continuity there exists a subregion of $D$ on which $(\nabla \times \mathbf{F}) \neq \mathbf{0}$.
> In the subregion choose a circular disk whose normal $\mathbf{n}$ is parallel to $\nabla \times \mathbf{F}$.

\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS \neq 0
\]

which contradicts our first assumption.
Theorem

Suppose that $\mathbf{F}(x, y, z)$ has continuous first partial derivatives throughout a simply connected region $D$, then the following statements are equivalent.

1. $\mathbf{F}$ is conservative in $D$, i.e. $\mathbf{F} = \nabla f$.
2. $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in $D$.
3. $\mathbf{F}$ is irrotational in $D$, i.e. $\nabla \times \mathbf{F} = \mathbf{0}$ in $D$.
4. $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every simple closed curve in $D$. 
Example

Let $\mathbf{F}(x, y, z) = y^2\mathbf{i} + (2xy + e^{3z})\mathbf{j} + 3ye^{3z}\mathbf{k}$ and show that $\mathbf{F}$ is conservative.
Example

Let $\mathbf{F}(x, y, z) = y^2 \mathbf{i} + (2xy + e^{3z}) \mathbf{j} + 3ye^{3z} \mathbf{k}$ and show that $\mathbf{F}$ is conservative.

We can accomplish this by showing that $\nabla \times \mathbf{F} = 0$.

$$\nabla \times \mathbf{F} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
y^2 & 2xy + e^{3z} & 3ye^{3z}
\end{vmatrix} = \langle 0, 0, 0 \rangle$$
Homework

- Read Section 14.8.
- Exercises: 1–33 odd