Surface Integrals
MATH 311, *Calculus III*

J. Robert Buchanan

Department of Mathematics

Summer 2015
Background

In calculus we have studied several types of definite integrals:
In calculus we have studied several types of definite integrals:

1. line integral
In calculus we have studied several types of definite integrals:

1. line integral
2. double integral
Background

In calculus we have studied several types of definite integrals:

1. line integral
2. double integral
3. triple integral
In calculus we have studied several types of definite integrals:

1. line integral
2. double integral
3. triple integral

Today we study the **surface integral**.
Motivation

Suppose we plot the surface $z = x + y^2$ and we color each point on the surface according to the value of its $y$-coordinate. The surface will appear to have stripes perpendicular to the $y$-axis.
Question

Keeping the previous surface plot in mind, what does it mean to integrate \( f(x, y, z) = y \) over the surface defined as \( z = x + y^2 \)?
Question

Keeping the previous surface plot in mind, what does it mean to integrate \( f(x, y, z) = y \) over the surface defined as \( z = x + y^2 \)?

Remarks:

- this is not the same as finding the volume beneath the surface,
  \[
  \int\int_{R} (x + y^2) \, dA
  \]
- it is not integrating \( f(x, y, z) \) over the volume beneath the surface,
  \[
  \int\int\int_{Q} y \, dV
  \]
- nor is it integrating \( f(x, y, z) \) over the projection of the surface in the xy-plane
  \[
  \int\int_{R} y \, dA
  \]
Riemann Sum

Given a function \( g(x, y, z) \) defined on a surface \( S \subset \mathbb{R}^3 \):

1. let \( P = \{S_1, S_2, \ldots, S_n\} \) be a partition of \( S \),
2. for each \( i = 1, 2, \ldots, n \) select an evaluation point \( (x_i, y_i, z_i) \in S_i \),
3. let \( \Delta S_i \) be the surface area of \( S_i \),

then a Riemann sum approximation to the surface integral is

\[
\sum_{i=1}^{n} g(x_i, y_i, z_i) \Delta S_i.
\]
Each partition element $S_i$ has a **diameter** defined as the maximum distance between any two points in $S_i$.

Define the norm of the partition $\|P\|$ to be the maximum of all the diameters of the partition elements.
**Surface Integral**

**Definition**

The **surface integral** of a function \( g(x, y, z) \) over a surface \( S \subset \mathbb{R}^3 \) is denoted

\[
\iint_S g(x, y, z) \, dS
\]

and is defined as

\[
\iint_S g(x, y, z) \, dS = \lim_{\|P\| \to 0} \sum_{i=1}^{n} g(x_i, y_i, z_i) \Delta S_i
\]

provided the limit exists and is the same for all choices of the evaluation points \((x_i, y_i, z_i)\).
Surface Integral

Definition

The **surface integral** of a function $g(x, y, z)$ over a surface $S \subset \mathbb{R}^3$ is denoted

$$\iint_S g(x, y, z) \, dS$$

and is defined as

$$\iint_S g(x, y, z) \, dS = \lim_{\|P\| \to 0} \sum_{i=1}^{n} g(x_i, y_i, z_i) \Delta S_i$$

provided the limit exists and is the same for all choices of the evaluation points $(x_i, y_i, z_i)$.

**Remark:** if $g(x, y, z) = 1$ then the surface integral produces the surface area of $S$. 
Suppose the surface $S$ can be described as $z = f(x, y)$ (similar derivations follow if $y = h(x, z)$ or $x = k(y, z)$), then the integrand

$$g(x, y, z) = g(x, y, f(x, y))$$

is a function to $x$ and $y$ only.

Let $R$ be the projection of $S$ into the $xy$-plane and let $P = \{R_1, R_2, \ldots, R_n\}$ be an inner partition of $R$.

For each $i = 1, 2, \ldots, n$, let $(x_i, y_i)$ be the point in $R_i$ closest to the origin. The evaluation points for the Riemann sum will be $(x_i, y_i, f(x_i, y_i))$. 
The surface area element $S_i$ can be approximated by a tangent parallelogram $T_i$ to the surface at the point $(x_i, y_i, f(x_i, y_i))$. The vectors $\langle 1, 0, f_x(x_i, y_i, z_i) \rangle$ and $\langle 0, 1, f_y(x_i, y_i, z_i) \rangle$ are parallel to two of the adjacent sides of $T_i$.

A normal vector to the surface (and $T_i$) at $(x_i, y_i, z_i)$ is

$$n_i = \langle 0, 1, f_y \rangle \times \langle 1, 0, f_x \rangle = \langle f_x, f_y, -1 \rangle.$$ 

The area of $T_i$ is

$$\|n_i\| = \sqrt{(f_x)^2 + (f_y)^2 + 1} \Delta A_i$$

where $\Delta A_i$ is the area of $R_i$. 

Evaluating a Surface Integral (2 of 3)
Evaluating a Surface Integral (3 of 3)

Theorem (Evaluation Theorem)

If the surface $S$ is given by $z = f(x, y)$ for $(x, y)$ in the region $R \subset \mathbb{R}^2$, where $f$ has continuous first partial derivatives, then

$$
\iint_S g(x, y, z) \, dS = \iint_R g(x, y, f(x, y)) \sqrt{(f_x)^2 + (f_y)^2 + 1} \, dA.
$$

Proof.

$$
\begin{align*}
\iint_S g(x, y, z) \, dS &= \lim_{\|P\| \to 0} \sum_{i=1}^{n} g(x_i, y_i, f(x_i, y_i)) \sqrt{(f_x)^2 + (f_y)^2 + 1} \Delta A_i \\
&= \iint_R g(x, y, f(x, y)) \sqrt{(f_x)^2 + (f_y)^2 + 1} \, dA
\end{align*}
$$
Example (1 of 2)

Evaluate the surface integral \( \iint_S y \, dS \) where \( S \) is the surface \( z = x + y^2 \) for \( 0 \leq x \leq 1 \) and \( 0 \leq y \leq 2 \).
Example (2 of 2)

\[
\int\int_S y \, dS = \int\int_R y \sqrt{(1)^2 + (2y)^2 + 1} \, dA
\]
\[
= \int_0^2 \int_0^1 y \sqrt{2 + 4y^2} \, dx \, dy
\]
\[
= \int_0^2 y \sqrt{2 + 4y^2} \, dy
\]
\[
= \frac{1}{8} \int_2^{18} u^{1/2} \, du
\]
\[
= \frac{1}{12} u^{3/2} \bigg|_2^{18}
\]
\[
= \frac{13\sqrt{2}}{3}
\]
Example (1 of 3)

Evaluate the surface integral \( \int \int_S x^2 \, dS \) where \( S \) is the unit sphere centered at the origin.
Example (2 of 3)

Let the western hemisphere of the unit sphere be the surface where
\[ x = \sqrt{1 - y^2 - z^2}. \]

\[
\iint_S x^2 \, dS = 2 \iint_{S'} x^2 \, dS \quad (S'\text{: western hemisphere})
\]

\[
= 2 \iint_R (1 - y^2 - z^2) \sqrt{\frac{y^2}{1 - y^2 - z^2} + \frac{z^2}{1 - y^2 - z^2} + 1} \, dA
\]

\[
= 2 \iint_R \frac{1 - y^2 - z^2}{\sqrt{1 - x^2 - y^2}} \, dA
\]

\[
= 2 \iint_R \sqrt{1 - x^2 - y^2} \, dA
\]
Example (3 of 3)

Evaluate the double integral in polar coordinates.

\[ \int \int_S x^2 \, dS = 2 \int \int_R \sqrt{1 - x^2 - y^2} \, dA \]

\[ = 2 \int_0^{2\pi} \int_0^1 \sqrt{1 - r^2} r \, dr \, d\theta \]

\[ = 4\pi \int_0^1 \sqrt{1 - r^2} r \, dr \]

\[ = 4\pi \int_1^0 u^{1/2} \left( -\frac{1}{2} \right) \, du \]

\[ = 2\pi \int_0^1 u^{1/2} \, du \]

\[ = \frac{4\pi}{3} \]
Flux and Flux Integrals

Suppose $S$ is a surface in $\mathbb{R}^3$ and $F(x, y, z)$ is a vector field defined in $\mathbb{R}^3$.

If $n$ is a unit normal vector to $S$ then $F \cdot n$ is a scalar function which can be thought of as the \textbf{component of $F$ perpendicular to} $S$. 
Suppose $S$ is a surface in $\mathbb{R}^3$ and $F(x, y, z)$ is a vector field defined in $\mathbb{R}^3$.

If $n$ is a unit normal vector to $S$ then $F \cdot n$ is a scalar function which can be thought of as the component of $F$ perpendicular to $S$.

**Definition**

The surface integral $\iint_S F \cdot n \, dS$ is called the flux integral of $F$ over $S$ or simply the flux of $F$ over $S$. 
Suppose $\mathbf{F}(x, y, z) = yi + xj + zk$ and $S$ is the boundary of the region enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$. Calculate the flux of $\mathbf{F}$ over $S$. 
Example (2 of 4)

On the plane $z = 0$ the downward normal is $\mathbf{n} = \langle 0, 0, -1 \rangle$.

\[
\iint_{S_1} \mathbf{F}(x, y, z) \cdot \mathbf{n} \, dS = \iint_{S_1} \langle y, x, 0 \rangle \cdot \langle 0, 0, -1 \rangle \, dS \\
= \iint_{S_1} 0 \, dS \\
= 0
\]
Example (3 of 4)

On the paraboloid $z + x^2 + y^2 = 1$ the upward unit normal is

$$n = \frac{\nabla (z + x^2 + y^2)}{\|\nabla (z + x^2 + y^2)\|} = \frac{\langle 2x, 2y, 1 \rangle}{\sqrt{4x^2 + 4y^2 + 1}}.$$

$$\int \int_{S_2} F(x, y, z) \cdot n \, dS$$

$$= \int \int_{S_2} \langle y, x, 1 - x^2 - y^2 \rangle \cdot \frac{\langle 2x, 2y, 1 \rangle}{\sqrt{4x^2 + 4y^2 + 1}} dS$$

$$= \int \int_{S_2} \frac{2xy + 2xy + 1 - x^2 - y^2}{\sqrt{4x^2 + 4y^2 + 1}} dS$$

$$= \int \int_{R} \frac{4xy + 1 - x^2 - y^2}{\sqrt{4x^2 + 4y^2 + 1}} \sqrt{4x^2 + 4y^2 + 1} \, dA$$

$$= \int \int_{R} 4xy + 1 - x^2 - y^2 \, dA$$
Example (4 of 4)

\[
\begin{align*}
\iiint_{S_2} F(x, y, z) \cdot n \, dS &= \iiint_R 4xy + 1 - x^2 - y^2 \, dA \\
&= \int_0^{2\pi} \int_0^1 (4r^2 \sin \theta \cos \theta + 1 - r^2) r \, dr \, d\theta \\
&= 2\pi \int_0^1 (r - r^3) \, dr \\
&= \frac{\pi}{2}
\end{align*}
\]

Thus the total flux is

\[
\iiint_S F \cdot n \, dS = \iiint_{S_1} F \cdot n \, dS + \iiint_{S_2} F \cdot n \, dS = 0 + \frac{\pi}{2} = \frac{\pi}{2}.
\]
Example (1 of 3)

Suppose \( \mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{j} + x\mathbf{k} \) and \( S \) is the boundary of the unit sphere. Calculate the flux of \( \mathbf{F} \) over \( S \).
Example (2 of 3)

The unit outward normal on the sphere $x^2 + y^2 + z^2 = 1$ is

$$\mathbf{n} = \frac{\nabla (x^2 + y^2 + z^2)}{\|\nabla (x^2 + y^2 + z^2)\|} = \frac{\langle 2x, 2y, 2z \rangle}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \langle x, y, z \rangle$$

when $x^2 + y^2 + z^2 = 1$.

$$\int \int \int_{S} \mathbf{F} \cdot \mathbf{n} dS = 2 \int \int \int_{S'} \mathbf{F} \cdot \mathbf{n} dS \quad (S': \text{northern hemisphere})$$

$$= 2 \int \int \int_{S'} \langle z, y, x \rangle \cdot \langle x, y, z \rangle dS$$

$$= 2 \int \int \int_{S'} (2xz + y^2) dS$$

$$= 2 \int \int_{R} \frac{2xz + y^2}{\sqrt{1 - x^2 - y^2}} dA$$
Example (3 of 3)

$$\int \int_S \mathbf{F} \cdot \mathbf{n} \, dS = 2 \int \int_R \frac{2xz + y^2}{\sqrt{1 - x^2 - y^2}} \, dA$$

$$= 2 \int_0^{2\pi} \int_0^1 \frac{2r^2 \cos \theta \sqrt{1 - r^2} + r^3 \sin^2 \theta}{\sqrt{1 - r^2}} \, dr \, d\theta$$

$$= 2 \int_0^{2\pi} \int_0^1 \frac{r^3 \sin^2 \theta}{\sqrt{1 - r^2}} \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 \frac{r^3 (1 - \cos 2\theta)}{\sqrt{1 - r^2}} \, dr \, d\theta$$

$$= 2\pi \int_0^1 \frac{r^3}{\sqrt{1 - r^2}} \, dr$$

$$= \frac{4\pi}{3}$$
Interpretation of Flux

Remarks:

- If $F$ represents the direction of flow of a fluid through a thin membrane represented by $S$, then the flux

$$\iint_S F \cdot n \, dS$$

is the volume of fluid passing through $S$. 

- If the fluid has density given by the scalar function $\rho(x, y, z)$ then

$$\iint_S \rho F \cdot n \, dS$$

is the mass of fluid passing through $S$. 

Remarks:

- If $\mathbf{F}$ represents the direction of flow of a fluid through a thin membrane represented by $S$, then the flux

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS$$

is the volume of fluid passing through $S$.

- If the fluid has density given by the scalar function $\rho(x, y, z)$ then

$$\iint_S \rho \mathbf{F} \cdot \mathbf{n} \, dS$$

is the mass of fluid passing through $S$. 
Application: Heat Flow

Definition
If the temperature of an object at point \((x, y, z)\) is given by \(u(x, y, z)\), then the heat flow is the vector field

\[
F = -\kappa \nabla u
\]

where \(\kappa\) is the thermal conductivity of the substance. The rate of heat flow across a surface \(S\) is given by the surface integral

\[
\iint_S F \cdot n \, dS = -\kappa \iint_S \nabla u \cdot n \, dS.
\]

where \(n\) is the unit normal vector to the surface.
Example

The temperature $u$ in a metal ball is proportional to the square of the distance from the center of the ball. Find the rate of heat flow across a sphere $S$ of radius $a > 0$ with center at the center of the ball.
Let $\kappa$ be the thermal conductivity of the metal.

$$u(x, y, z) = C(x^2 + y^2 + z^2)$$

$$F(x, y, z) = -\kappa \nabla u = -2\kappa C \langle x, y, z \rangle$$

$$n = \frac{1}{a} \langle x, y, z \rangle$$

Thus

$$\int \int_S F \cdot n \, dS = \int \int_S -\frac{2\kappa C}{a} (x^2 + y^2 + z^2) \, dS$$

$$= \int \int_S -\frac{2\kappa C}{a} a^2 \, dS$$

$$= -2\kappa Ca \int \int_S 1 \, dS = -8\kappa C\pi a^3.$$
Application: Electrostatics

Definition
If \( \mathbf{E} \) is an electric field, then the surface integral

\[
\iint_S \mathbf{E} \cdot \mathbf{n} \, dS
\]

is called the **electric flux** of \( \mathbf{E} \) through surface \( S \). \textbf{Gauss’s Law} states that the net charge enclosed by a closed surface \( S \) is

\[
Q = \epsilon_0 \iint_S \mathbf{E} \cdot \mathbf{n} \, dS
\]

where the constant \( \epsilon_0 \approx 8.8542 \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2 \).
Example

Find the charge contained in the solid hemisphere $x^2 + y^2 + z^2 \leq a^2$ with $z \geq 0$ if the electric field is $\mathbf{E}(x, y, z) = \langle x, y, 2z \rangle$. 
On the surface $S_1$, $x^2 + y^2 \leq a^2$ and $z = 0$ the unit outward normal is $\mathbf{n} = \langle 0, 0, -1 \rangle$.

\[
Q = \epsilon_0 \int\int_{S_1} \mathbf{E} \cdot \mathbf{n} \, dS
\]

\[
= \epsilon_0 \int\int_{S_1} \langle x, y, 0 \rangle \cdot \langle 0, 0, -1 \rangle \, dS
\]

\[
= \epsilon_0 \int\int_{S_1} 0 \, dS
\]

\[
= 0
\]
Solution (2 of 3)

On the surface $S_2$, $x^2 + y^2 + z^2 \leq a^2$ and $z > 0$ the unit outward normal is $\mathbf{n} = \frac{1}{a} \langle x, y, z \rangle$.

\[
Q = \varepsilon_0 \int \int_{S_2} \mathbf{E} \cdot \mathbf{n} \, dS
\]

\[
= \frac{\varepsilon_0}{a} \int \int_{S_2} \langle x, y, 2z \rangle \cdot \langle x, y, z \rangle \, dS
\]

\[
= \frac{\varepsilon_0}{a} \int \int_{S_2} (x^2 + y^2 + 2z^2) \, dS
\]

\[
= \frac{\varepsilon_0}{a} \int \int_{S_2} (2a^2 - x^2 - y^2) \, dS
\]

\[
= \frac{\varepsilon_0}{a} \int \int_{R} (2a^2 - x^2 - y^2) \frac{a}{\sqrt{a^2 - x^2 - y^2}} \, dA
\]

\[
= \varepsilon_0 \int \int_{R} \frac{2a^2 - x^2 - y^2}{\sqrt{a^2 - x^2 - y^2}} \, dA
\]
Solution (3 of 3)

\[ Q = \epsilon_0 \int_0^{2\pi} \int_0^a \frac{2a^2 r - r^3}{\sqrt{a^2 - r^2}} \, dr \, d\theta \]
\[ = 2\pi \epsilon_0 \int_0^a \frac{2a^2 r - r^3}{\sqrt{a^2 - r^2}} \, dr \]
\[ = 2\pi \epsilon_0 \int_0^a \left[ r \sqrt{a^2 - r^2} + \frac{a^2 r}{\sqrt{a^2 - r^2}} \right] \, dr \]
\[ = 2\pi \epsilon_0 \left[ -\frac{1}{3} (a^2 - r^2)^{3/2} - a^2 \sqrt{a^2 - r^2} \right]_0^a \]
\[ = 2\pi \epsilon_0 \left[ \frac{a^3}{3} + a^3 \right] \]
\[ = \frac{8}{3} \pi a^3 \epsilon_0 \]
Homework

- Read Section 14.6.
- Exercises: 19–69 odd