Surface Integrals
MATH 311, *Calculus III*

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1. line integral
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Background

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2. double integral
3. triple integral
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1. line integral
2. double integral
3. triple integral

Today we study the **surface integral**.
Suppose we plot the surface $z = x + y^2$ and we color each point on the surface according to the value of its $y$-coordinate. The surface will appear to have stripes perpendicular to the $y$-axis.
Question

Keeping the previous surface plot in mind, what does it mean to integrate $f(x, y, z) = y$ over the surface defined as $z = x + y^2$?
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Keeping the previous surface plot in mind, what does it mean to integrate \( f(x, y, z) = y \) over the surface defined as \( z = x + y^2 \)?

Remarks:

▶ this is not the same as finding the volume beneath the surface,
\[
\iiint_{R} (x + y^2) \, dA
\]

▶ it is not integrating \( f(x, y, z) \) over the volume beneath the surface,
\[
\iiint_{Q} y \, dV
\]

▶ nor is it integrating \( f(x, y, z) \) over the projection of the surface in the \( xy \)-plane
\[
\iint_{R} y \, dA
\]
Riemann Sum

Given a function $g(x, y, z)$ defined on a surface $S \subset \mathbb{R}^3$:

1. let $P = \{S_1, S_2, \ldots, S_n\}$ be a partition of $S$,
2. for each $i = 1, 2, \ldots, n$ select an evaluation point $(x_i, y_i, z_i) \in S_i$,
3. let $\Delta S_i$ be the surface area of $S_i$,

then a Riemann sum approximation to the surface integral is

$$\sum_{i=1}^{n} g(x_i, y_i, z_i) \Delta S_i.$$
Each partition element $S_i$ has a **diameter** defined as the maximum distance between any two points in $S_i$.

Define the norm of the partition $\|P\|$ to be the maximum of all the diameters of the partition elements.
Surface Integral

Definition
The **surface integral** of a function \( g(x, y, z) \) over a surface \( S \subset \mathbb{R}^3 \) is denoted
\[
\iiint_S g(x, y, z) \, dS
\]
and is defined as
\[
\iiint_S g(x, y, z) \, dS = \lim_{\|P\| \to 0} \sum_{i=1}^{n} g(x_i, y_i, z_i) \Delta S_i
\]
provided the limit exists and is the same for all choices of the evaluation points \((x_i, y_i, z_i)\).
Surface Integral

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provided the limit exists and is the same for all choices of the evaluation points $(x_i, y_i, z_i)$.

Remark: if $g(x, y, z) = 1$ then the surface integral produces the surface area of $S$. 
Suppose the surface $S$ can be described as $z = f(x, y)$ (similar derivations follow if $y = h(x, z)$ or $x = k(y, z)$), then the integrand
\[ g(x, y, z) = g(x, y, f(x, y)) \]
is a function to $x$ and $y$ only.

Let $R$ be the projection of $S$ into the $xy$-plane and let $P = \{R_1, R_2, \ldots, R_n\}$ be an inner partition of $R$.

For each $i = 1, 2, \ldots, n$, let $(x_i, y_i)$ be the point in $R_i$ closest to the origin. The evaluation points for the Riemann sum will be $(x_i, y_i, f(x_i, y_i))$. 
The surface area element $S_i$ can be approximated by a tangent parallelogram $T_i$ to the surface at the point $(x_i, y_i, f(x_i, y_i))$.

The vectors $\langle 1, 0, f_x(x_i, y_i, z_i) \rangle$ and $\langle 0, 1, f_y(x_i, y_i, z_i) \rangle$ are parallel to two of the adjacent sides of $T_i$.

A normal vector to the surface (and $T_i$) at $(x_i, y_i, z_i)$ is

$$n_i = \langle 0, 1, f_y \rangle \times \langle 1, 0, f_x \rangle = \langle f_x, f_y, -1 \rangle.$$ 

The area of $T_i$ is

$$\|n_i\| = \sqrt{(f_x)^2 + (f_y)^2 + 1 \Delta A_i}$$

where $\Delta A_i$ is the area of $R_i$. 

Evaluating a Surface Integral (2 of 3)
Evaluating a Surface Integral (3 of 3)

**Theorem (Evaluation Theorem)**

If the surface $S$ is given by $z = f(x, y)$ for $(x, y)$ in the region $R \subset \mathbb{R}^2$, where $f$ has continuous first partial derivatives, then

$$\int\int_S g(x, y, z) \, dS = \int\int_R g(x, y, f(x, y)) \sqrt{(f_x)^2 + (f_y)^2 + 1} \, dA.$$

**Proof.**

$$\int\int_S g(x, y, z) \, dS$$

$$= \lim_{\|P\| \to 0} \sum_{i=1}^{n} g(x_i, y_i, f(x_i, y_i)) \sqrt{(f_x)^2 + (f_y)^2 + 1} \Delta A_i$$

$$= \int\int_R g(x, y, f(x, y)) \sqrt{(f_x)^2 + (f_y)^2 + 1} \, dA.$$
Evaluate the surface integral $\iint_S y \, dS$ where $S$ is the surface $z = x + y^2$ for $0 \leq x \leq 1$ and $0 \leq y \leq 2$. 
\[ \int\int_S y \, dS = \int\int_R y \sqrt{(1)^2 + (2y)^2 + 1} \, dA \]
\[ = \int_0^2 \int_0^1 y \sqrt{2 + 4y^2} \, dx \, dy \]
\[ = \int_0^2 y \sqrt{2 + 4y^2} \, dy \]
\[ = \frac{1}{8} \int_2^{18} u^{1/2} \, du \]
\[ = \frac{1}{12} u^{3/2} \bigg|_2^{18} \]
\[ = \frac{13\sqrt{2}}{3} \]
Evaluate the surface integral \( \int\int_S x^2 \, dS \) where \( S \) is the unit sphere centered at the origin.
Example (2 of 3)

Let the western hemisphere of the unit sphere be the surface where \( x = \sqrt{1 - y^2 - z^2} \).

\[
\int\int_S x^2 \, dS = 2 \int\int_{S'} x^2 \, dS \quad (S': \text{western hemisphere})
\]

\[
= 2 \int\int_R (1 - y^2 - z^2) \sqrt{\frac{y^2}{1 - y^2 - z^2} + \frac{z^2}{1 - y^2 - z^2} + 1} \, dA
\]

\[
= 2 \int\int_R \frac{1 - y^2 - z^2}{\sqrt{1 - x^2 - y^2}} \, dA
\]

\[
= 2 \int\int_R \sqrt{1 - x^2 - y^2} \, dA
\]
Example (3 of 3)

Evaluate the double integral in polar coordinates.

\[ \int \int_S x^2 \, dS = 2 \int \int_R \sqrt{1 - x^2 - y^2} \, dA \]

\[ = 2 \int_0^{2\pi} \int_0^1 \sqrt{1 - r^2} \, r \, dr \, d\theta \]

\[ = 4\pi \int_0^1 \sqrt{1 - r^2} \, r \, dr \]

\[ = 4\pi \int_0^1 u^{1/2} \left( -\frac{1}{2} \right) \, du \]

\[ = 2\pi \int_0^1 u^{1/2} \, du \]

\[ = \frac{4\pi}{3} \]
Flux and Flux Integrals

Suppose $S$ is a surface in $\mathbb{R}^3$ and $\mathbf{F}(x, y, z)$ is a vector field defined in $\mathbb{R}^3$.

If $\mathbf{n}$ is a unit normal vector to $S$ then $\mathbf{F} \cdot \mathbf{n}$ is a scalar function which can be thought of as the component of $\mathbf{F}$ perpendicular to $S$. 
Suppose $S$ is a surface in $\mathbb{R}^3$ and $\mathbf{F}(x, y, z)$ is a vector field defined in $\mathbb{R}^3$.

If $\mathbf{n}$ is a unit normal vector to $S$ then $\mathbf{F} \cdot \mathbf{n}$ is a scalar function which can be thought of as the component of $\mathbf{F}$ perpendicular to $S$.

**Definition**

The surface integral $\iint_S \mathbf{F} \cdot \mathbf{n} \, dS$ is called the flux integral of $\mathbf{F}$ over $S$ or simply the flux of $\mathbf{F}$ over $S$. 
Example (1 of 4)

Suppose \( \mathbf{F}(x, y, z) = yi + xj + zk \) and \( S \) is the boundary of the region enclosed by the paraboloid \( z = 1 - x^2 - y^2 \) and the plane \( z = 0 \). Calculate the flux of \( \mathbf{F} \) over \( S \). 
Example (2 of 4)

On the plane $z = 0$ the downward normal is $\mathbf{n} = \langle 0, 0, -1 \rangle$.

$$\int\int_{S_1} \mathbf{F}(x, y, z) \cdot \mathbf{n} \, dS = \int\int_{S_1} \langle y, x, 0 \rangle \cdot \langle 0, 0, -1 \rangle \, dS$$

$$= \int\int_{S_1} 0 \, dS$$

$$= 0$$
Example (3 of 4)

On the paraboloid \( z + x^2 + y^2 = 1 \) the upward unit normal is

\[
\mathbf{n} = \frac{\nabla(z + x^2 + y^2)}{\|\nabla(z + x^2 + y^2)\|} = \frac{\langle 2x, 2y, 1 \rangle}{\sqrt{4x^2 + 4y^2 + 1}}.
\]

\[
\int \int_{S_2} \mathbf{F}(x, y, z) \cdot \mathbf{n} \, dS
\]

\[
= \int \int_{S_2} \langle y, x, 1 - x^2 - y^2 \rangle \cdot \frac{\langle 2x, 2y, 1 \rangle}{\sqrt{4x^2 + 4y^2 + 1}} \, dS
\]

\[
= \int \int_{S_2} \frac{2xy + 2xy + 1 - x^2 - y^2}{\sqrt{4x^2 + 4y^2 + 1}} \, dS
\]

\[
= \int \int_{R} \frac{4xy + 1 - x^2 - y^2}{\sqrt{4x^2 + 4y^2 + 1}} \sqrt{4x^2 + 4y^2 + 1} \, dA
\]

\[
= \int \int_{R} 4xy + 1 - x^2 - y^2 \, dA
\]
Example (4 of 4)

\[ \int \int_{S_2} \mathbf{F}(x, y, z) \cdot \mathbf{n} \, dS \]

\[ = \int \int_{R} 4xy + 1 - x^2 - y^2 \, dA \]

\[ = \int_{0}^{2\pi} \int_{0}^{1} (4r^2 \sin \theta \cos \theta + 1 - r^2)r \, dr \, d\theta \]

\[ = 2\pi \int_{0}^{1} (r - r^3) \, dr \]

\[ = \frac{\pi}{2} \]

Thus the total flux is

\[ \int \int_{S} \mathbf{F} \cdot \mathbf{n} \, dS = \int \int_{S_1} \mathbf{F} \cdot \mathbf{n} \, dS + \int \int_{S_2} \mathbf{F} \cdot \mathbf{n} \, dS = 0 + \frac{\pi}{2} = \frac{\pi}{2}. \]
Suppose \( \mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{j} + x\mathbf{k} \) and \( S \) is the boundary of the unit sphere. Calculate the flux of \( \mathbf{F} \) over \( S \).
Example (2 of 3)

The unit outward normal on the sphere $x^2 + y^2 + z^2 = 1$ is

$$
\mathbf{n} = \frac{\nabla(x^2 + y^2 + z^2)}{\|\nabla(x^2 + y^2 + z^2)\|} = \frac{\langle 2x, 2y, 2z \rangle}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \langle x, y, z \rangle
$$

when $x^2 + y^2 + z^2 = 1$.

$$
\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = 2 \iint_{S'} \mathbf{F} \cdot \mathbf{n} \, dS \quad (S': \text{northern hemisphere})
$$

$$
= 2 \iint_{S'} \langle z, y, x \rangle \cdot \langle x, y, z \rangle \, dS
$$

$$
= 2 \iint_{S'} (2xz + y^2) \, dS
$$

$$
= 2 \iint_R \frac{2xz + y^2}{\sqrt{1 - x^2 - y^2}} \, dA
$$
Example (3 of 3)

\[ \iint_S \mathbf{F} \cdot \mathbf{n} \, dS = 2 \iint_R \frac{2xz + y^2}{\sqrt{1 - x^2 - y^2}} \, dA \]

\[ = 2 \int_0^{2\pi} \int_0^1 2r^2 \cos \theta \sqrt{1 - r^2} + r^3 \sin^2 \theta \frac{1}{\sqrt{1 - r^2}} \, dr \, d\theta \]

\[ = 2 \int_0^{2\pi} \int_0^1 \frac{r^3 \sin^2 \theta}{\sqrt{1 - r^2}} \, dr \, d\theta \]

\[ = \int_0^{2\pi} \int_0^1 \frac{r^3 (1 - \cos 2\theta)}{\sqrt{1 - r^2}} \, dr \, d\theta \]

\[ = 2\pi \int_0^1 \frac{r^3}{\sqrt{1 - r^2}} \, dr \]

\[ = \frac{4\pi}{3} \]
Remarks:

If $\mathbf{F}$ represents the direction of flow of a fluid through a thin membrane represented by $S$, then the flux

$$
\iint_S \mathbf{F} \cdot \mathbf{n} \, dS
$$

is the volume of fluid passing through $S$. 

If the fluid has density given by the scalar function $\rho(x, y, z)$ then

$$
\iint_S \rho \mathbf{F} \cdot \mathbf{n} \, dS
$$

is the mass of fluid passing through $S$. 
Remarks:

- If \( \mathbf{F} \) represents the direction of flow of a fluid through a thin membrane represented by \( S \), then the flux

\[
\iint_S \mathbf{F} \cdot \mathbf{n} \, dS
\]

is the volume of fluid passing through \( S \).

- If the fluid has density given by the scalar function \( \rho(x, y, z) \) then

\[
\iint_S \rho \mathbf{F} \cdot \mathbf{n} \, dS
\]

is the mass of fluid passing through \( S \).
Application: Heat Flow

Definition
If the temperature of an object at point \((x, y, z)\) is given by \(u(x, y, z)\), then the heat flow is the vector field

\[
F = -\kappa \nabla u
\]

where \(\kappa\) is the thermal conductivity of the substance. The rate of heat flow across a surface \(S\) is given by the surface integral

\[
\iint_S F \cdot n \, dS = -\kappa \iint_S \nabla u \cdot n \, dS.
\]

where \(n\) is the unit normal vector to the surface.
Example

The temperature $u$ in a metal ball is proportional to the square of the distance from the center of the ball. Find the rate of heat flow across a sphere $S$ of radius $a > 0$ with center at the center of the ball.
Solution

Let $\kappa$ be the thermal conductivity of the metal.

\[ u(x, y, z) = C(x^2 + y^2 + z^2) \]
\[ F(x, y, z) = -\kappa \nabla u = -2\kappa C \langle x, y, z \rangle \]
\[ n = \frac{1}{a} \langle x, y, z \rangle \]

Thus

\[
\int \int_S F \cdot n \, dS = \int \int_S -\frac{2\kappa C}{a} (x^2 + y^2 + z^2) \, dS
\]
\[
= \int \int_S -\frac{2\kappa C}{a} a^2 \, dS
\]
\[
= -2\kappa Ca \int \int_S 1 \, dS = -8\kappa C\pi a^3.
\]
Application: Electrostatics

Definition
If $\mathbf{E}$ is an electric field, then the surface integral

$$\int \int_S \mathbf{E} \cdot \mathbf{n} \, dS$$

is called the electric flux of $\mathbf{E}$ through surface $S$. Gauss’s Law states that the net charge enclosed by a closed surface $S$ is

$$Q = \varepsilon_0 \int \int_S \mathbf{E} \cdot \mathbf{n} \, dS$$

where the constant $\varepsilon_0 \approx 8.8542 \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2$. 
Example

Find the charge contained in the solid hemisphere \( x^2 + y^2 + z^2 \leq a^2 \) with \( z \geq 0 \) if the electric field is \( \mathbf{E}(x, y, z) = \langle x, y, 2z \rangle \).
Solution (1 of 3)

On the surface $S_1$, $x^2 + y^2 \leq a^2$ and $z = 0$ the unit outward normal is $n = \langle 0, 0, -1 \rangle$.

$$Q = \varepsilon_0 \iiint_{S_1} E \cdot n \, dS$$

$$= \varepsilon_0 \iiint_{S_1} \langle x, y, 0 \rangle \cdot \langle 0, 0, -1 \rangle \, dS$$

$$= \varepsilon_0 \iiint_{S_1} 0 \, dS$$

$$= 0$$
Solution (2 of 3)

On the surface $S_2$, $x^2 + y^2 + z^2 \leq a^2$ and $z > 0$ the unit outward normal is $n = \frac{1}{a} \langle x, y, z \rangle$.

\[
Q = \epsilon_0 \int \int_{S_2} E \cdot n \, dS
\]

\[
= \frac{\epsilon_0}{a} \int \int_{S_2} \langle x, y, 2z \rangle \cdot \langle x, y, z \rangle \, dS
\]

\[
= \frac{\epsilon_0}{a} \int \int_{S_2} (x^2 + y^2 + 2z^2) \, dS
\]

\[
= \frac{\epsilon_0}{a} \int \int_{S_2} (2a^2 - x^2 - y^2) \, dS
\]

\[
= \frac{\epsilon_0}{a} \int \int_{R} (2a^2 - x^2 - y^2) \frac{a}{\sqrt{a^2 - x^2 - y^2}} \, dA
\]

\[
= \epsilon_0 \int \int_{R} \frac{2a^2 - x^2 - y^2}{\sqrt{a^2 - x^2 - y^2}} \, dA
\]
Solution (3 of 3)

\[ Q = \epsilon_0 \int_0^{2\pi} \int_0^a \frac{2a^2 r - r^3}{\sqrt{a^2 - r^2}} \, dr \, d\theta \]

\[ = 2\pi \epsilon_0 \int_0^a \frac{2a^2 r - r^3}{\sqrt{a^2 - r^2}} \, dr \]

\[ = 2\pi \epsilon_0 \left[ r\sqrt{a^2 - r^2} + \frac{a^2 r}{\sqrt{a^2 - r^2}} \right] \bigg|_0^a \]

\[ = 2\pi \epsilon_0 \left[ -\frac{1}{3} (a^2 - r^2)^{3/2} - a^2 \sqrt{a^2 - r^2} \right] \bigg|_0^a \]

\[ = 2\pi \epsilon_0 \left[ \frac{a^3}{3} + a^3 \right] \]

\[ = \frac{8}{3} \pi a^3 \epsilon_0 \]
Homework

▶ Read Section 14.6.
▶ Exercises: 19–69 odd