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**Q:** How do we create a moving coordinate system?
When an observer is traveling along with a moving point, for example the passengers in an airplane, it can be useful to have a right-handed coordinate system travel with the observer.

**Q:** How do we create a moving coordinate system?

**A:** We already have the unit tangent vector

\[ T(t) = \frac{r'(t)}{\|r'(t)\|} \]

which is in the same direction as the motion. We need only two more vectors perpendicular to it.
Definition
The *principal unit normal vector* $\mathbf{N}(t)$ is a unit vector having the same direction as $\mathbf{T}'(t)$ and is defined by

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}.$$
Principal Unit Normal Vector

Definition
The principal unit normal vector $\mathbf{N}(t)$ is a unit vector having the same direction as $\mathbf{T}'(t)$ and is defined by

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}.$$ 

Note: $\mathbf{N}(t) \neq \frac{\mathbf{r}''(t)}{\|\mathbf{r}''(t)\|}$
Direction of $\mathbf{N}(t)$

By the Chain Rule, $\mathbf{T}'(t) = \frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds} \frac{ds}{dt}$ so

$$
\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{\frac{d\mathbf{T}}{ds}}{\|\frac{d\mathbf{T}}{ds}\|} \frac{ds}{dt}
$$

$$
= \frac{\frac{d\mathbf{T}}{ds}}{\|\frac{d\mathbf{T}}{ds}\|} (\text{since } \|\mathbf{r}'(t)\| = \frac{ds}{dt} > 0)
$$

$$
= \frac{1}{\kappa} \frac{d\mathbf{T}}{ds} (\text{since } \kappa = \|\frac{d\mathbf{T}}{ds}\|)
$$

$\frac{d\mathbf{T}}{ds}$ points in the direction $\mathbf{T}$ is turning and thus $\mathbf{N}(t)$ always points to the concave side of the curve.
Illustration
Example (1 of 2)

Find the unit tangent and principal unit normal vectors for the following vector-valued function.

\[
\mathbf{r}(t) = \langle t, t^3 \rangle
\]
Example (1 of 2)

Find the unit tangent and principal unit normal vectors for the following vector-valued function.

\[ r(t) = \langle t, t^3 \rangle \]

\[ T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{1}{\sqrt{1 + 9t^4}} \langle 1, 3t^2 \rangle \]
Example (1 of 2)

Find the unit tangent and principal unit normal vectors for the following vector-valued function.

\[ r(t) = \langle t, t^3 \rangle \]

\[ T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{1}{\sqrt{1 + 9t^4}} \langle 1, 3t^2 \rangle \]

\[ T'(t) = \frac{6t}{(1 + 9t^4)^{3/2}} \langle -3t^2, 1 \rangle \]

Remark: in \( \mathbb{V}^2 \), vectors \( \langle a, b \rangle \) and \( \langle -b, a \rangle \) are always orthogonal.
Example (1 of 2)

Find the unit tangent and principal unit normal vectors for the following vector-valued function.

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\[ T'(t) = \frac{6t}{(1 + 9t^4)^{3/2}} \langle -3t^2, 1 \rangle \]

\[ N(t) = \frac{T'(t)}{\|T'(t)\|} = \frac{1}{\sqrt{1 + 9t^4}} \langle -3t^2, 1 \rangle \]

**Remark:** in \( V_2 \), vectors \( \langle a, b \rangle \) and \( \langle -b, a \rangle \) are always orthogonal.
Example (2 of 2)

Find the unit tangent and principal unit normal vectors for the following vector-valued function.

\[ r(t) = \langle \sin t, \cos t, \cos t \rangle \]
Example (2 of 2)

Find the unit tangent and principal unit normal vectors for the following vector-valued function.

\[ r(t) = \langle \sin t, \cos t, \cos t \rangle \]

\[
T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{\sqrt{2}}{\sqrt{3 - \cos 2t}} \langle \cos t, - \sin t, - \sin t \rangle
\]

\[
N(t) = \frac{\langle 2 \sin t, \cos t, \cos t \rangle}{\sqrt{3 - \cos 2t}}
\]
Example (2 of 2)

Find the unit tangent and principal unit normal vectors for the following vector-valued function.

\[ \mathbf{r}(t) = \langle \sin t, \cos t, \cos t \rangle \]

\[ \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\sqrt{2}}{\sqrt{3 - \cos 2t}} \langle \cos t, - \sin t, - \sin t \rangle \]

\[ \mathbf{T}'(t) = \frac{-2\sqrt{2}}{(3 - \cos 2t)^{3/2}} \langle 2 \sin t, \cos t, \cos t \rangle \]
Example (2 of 2)

Find the unit tangent and principal unit normal vectors for the following vector-valued function.

\[ \mathbf{r}(t) = \langle \sin t, \cos t, \cos t \rangle \]

\[ \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\| \mathbf{r}'(t) \|} = \frac{\sqrt{2}}{\sqrt{3 - \cos 2t}} \langle \cos t, -\sin t, -\sin t \rangle \]

\[ \mathbf{T}'(t) = \frac{-2\sqrt{2}}{(3 - \cos 2t)^{3/2}} \langle 2 \sin t, \cos t, \cos t \rangle \]

\[ \mathbf{N}(t) = -\frac{1}{\sqrt{3 - \cos 2t}} \langle 2 \sin t, \cos t, \cos t \rangle \]
Binormal Vector

So far we have a pair of orthogonal vectors defined along the path of motion. We can get the third orthogonal vector from the cross product.

**Definition**

The binormal vector \( B(t) \) is:

\[
B(t) = T(t) \times N(t).
\]

**Note:**
\[\|B(t)\|=\|T(t)\times N(t)\|=\|T(t)\|\|N(t)\|\sin\frac{\pi}{2}=1.\]
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**Definition**
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**Note:** $\|B(t)\| = \|T(t) \times N(t)\| = \|T(t)\| \|N(t)\| \sin \frac{\pi}{2} = 1$.

**Definition**
The triple of unit vectors $T(t)$, $N(t)$, and $B(t)$ forms a moving frame of reference called the **TNB frame** or **moving trihedral**.
Example

Find the binormal vector for the vector-valued function,

\[ r(t) = \langle \sin t, \cos t, \cos t \rangle \]
Example

Find the binormal vector for the vector-valued function,

\[ r(t) = \langle \sin t, \cos t, \cos t \rangle \]

We have already found:

\[ T(t) = \frac{\sqrt{2}}{\sqrt{3 - \cos 2t}} \langle \cos t, -\sin t, -\sin t \rangle \]

\[ N(t) = -\frac{1}{\sqrt{3 - \cos 2t}} \langle 2\sin t, \cos t, \cos t \rangle \]
Example

Find the binormal vector for the vector-valued function,

\[ \mathbf{r}(t) = \langle \sin t, \cos t, \cos t \rangle \]

We have already found:

\[ \mathbf{T}(t) = \frac{\sqrt{2}}{\sqrt{3 - \cos 2t}} \langle \cos t, -\sin t, -\sin t \rangle \]

\[ \mathbf{N}(t) = -\frac{1}{\sqrt{3 - \cos 2t}} \langle 2 \sin t, \cos t, \cos t \rangle \]

\[ \mathbf{B}(t) = \frac{1}{\sqrt{2}} \langle 0, 1, -1 \rangle \]
Osculating Circle

Definition
For a curve $C$ in the $xy$-plane, defined by the vector-valued function $\mathbf{r}(t)$, if the curvature $\kappa \neq 0$ when $t = t_0$, we define the **radius of curvature** to be $1/\kappa$. The **center of curvature** is the terminal point of the vector $\mathbf{r}(t) + \frac{1}{\kappa} \mathbf{N}(t)$ (when $\kappa \neq 0$). The **osculating circle** or **circle of curvature** is the circle whose center is the center of curvature and whose radius is the radius of curvature (when $\kappa \neq 0$).
Example

Find the center of curvature and the radius of curvature for the osculating circle to the curve \( r(t) = \langle t, t^3 \rangle \) when \( t = 1 \).
Example

Find the center of curvature and the radius of curvature for the osculating circle to the curve \( r(t) = \langle t, t^3 \rangle \) when \( t = 1 \).

\[
\kappa = \frac{|f''(t)|}{(1 + [f'(t)]^2)^{3/2}} = \frac{6|t|}{(1 + 9t^4)^{3/2}}
\]

\[
N(t) = \frac{1}{\sqrt{1 + 9t^4}} \langle -3t^2, 1 \rangle
\]
Example

Find the center of curvature and the radius of curvature for the osculating circle to the curve \( \mathbf{r}(t) = \langle t, t^3 \rangle \) when \( t = 1 \).

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\]

\[
\mathbf{N}(t) = \frac{1}{\sqrt{1 + 9t^4}} \langle -3t^2, 1 \rangle
\]

\[
\text{radius} = \frac{1}{\kappa(1)} = \frac{5\sqrt{10}}{3}
\]
Example

Find the center of curvature and the radius of curvature for the osculating circle to the curve \( r(t) = \langle t, t^3 \rangle \) when \( t = 1 \).

\[
\kappa = \frac{|f''(t)|}{(1 + [f'(t)]^2)^{3/2}} = \frac{6|t|}{(1 + 9t^4)^{3/2}}
\]

\[
N(t) = \frac{1}{\sqrt{1 + 9t^4}} \langle -3t^2, 1 \rangle
\]

radius \( = \frac{1}{\kappa(1)} = \frac{5\sqrt{10}}{3} \)

center \( = r(1) + \frac{1}{\kappa(1)}N(1) = \left(-4, \frac{8}{3}\right) \)
Tangential and Normal Components of Acceleration

The **unit tangent** and **principal unit normal** vectors can explain the forces which work to stabilize and destabilize an object as it moves on a path.
The unit tangent and principal unit normal vectors can explain the forces which work to stabilize and destabilize an object as it moves on a path.

The velocity of an object moving along a path described by $r(t)$ is

$$v(t) = r'(t) = \|r'(t)\| T(t) = \frac{ds}{dt} T(t).$$
The **unit tangent** and **principal unit normal** vectors can explain the forces which work to stabilize and destabilize an object as it moves on a path.

The velocity of an object moving along a path described by \( \mathbf{r}(t) \) is

\[
\mathbf{v}(t) = \mathbf{r}'(t) = \| \mathbf{r}'(t) \| \mathbf{T}(t) = \frac{ds}{dt} \mathbf{T}(t).
\]

The acceleration is given by

\[
\mathbf{a}(t) = \frac{d}{dt} \left( \frac{ds}{dt} \mathbf{T}(t) \right) = \frac{d^2s}{dt^2} \mathbf{T}(t) + \frac{ds}{dt} \mathbf{T}'(t).
\]
Tangential and Normal Components (continued)

\[
a(t) = \frac{d^2s}{dt^2} T(t) + \frac{ds}{dt} T'(t)
\]

\[
= \frac{d^2s}{dt^2} T(t) + \frac{ds}{dt} \| T'(t) \| N(t)
\]

\[
= \frac{d^2s}{dt^2} T(t) + \frac{ds}{dt} \left\| \frac{dT}{ds} \frac{ds}{dt} \right\| N(t)
\]

\[
= \frac{d^2s}{dt^2} T(t) + \left( \frac{ds}{dt} \right)^2 \left\| \frac{dT}{ds} \right\| N(t)
\]

\[
= \frac{d^2s}{dt^2} T(t) + \kappa \left( \frac{ds}{dt} \right)^2 N(t)
\]
Tangential and Normal Components (continued)

\[ a(t) = \frac{d^2 s}{dt^2} T(t) + \kappa \left( \frac{ds}{dt} \right)^2 N(t) \]

\( \frac{d^2 s}{dt^2} \): is called the **tangential component of acceleration** and is denoted \( a_T = \frac{d^2 s}{dt^2} \).

\( \kappa \left( \frac{ds}{dt} \right)^2 \): is called the **normal component of acceleration** and is denoted \( a_N = \kappa \left( \frac{ds}{dt} \right)^2 \).
Tangential and Normal Components (continued)

\[ \mathbf{a}(t) = \frac{d^2 s}{dt^2} \mathbf{T}(t) + \kappa \left( \frac{ds}{dt} \right)^2 \mathbf{N}(t) \]

\( \frac{d^2 s}{dt^2} \): is called the **tangential component of acceleration** and is denoted \( a_T = \frac{d^2 s}{dt^2} \).

\( \kappa \left( \frac{ds}{dt} \right)^2 \): is called the **normal component of acceleration** and is denoted \( a_N = \kappa \left( \frac{ds}{dt} \right)^2 \).

By Newton’s Second Law of Motion

\[ \mathbf{F}(t) = m \mathbf{a}(t) = m \frac{d^2 s}{dt^2} \mathbf{T}(t) + m \kappa \left( \frac{ds}{dt} \right)^2 \mathbf{N}(t). \]
Example (1 of 2)

Express the acceleration of the object traveling along the path described by \( r(t) = \langle \cos t, \sin t, \sin 3t \rangle \) in terms of the tangential and normal components.
Express the acceleration of the object traveling along the path described by \( \mathbf{r}(t) = \langle \cos t, \sin t, \sin 3t \rangle \) in terms of the tangential and normal components.

Strategy:

1. Find \( \frac{ds}{dt} = \|\mathbf{r}'(t)\| \).
2. Find \( a_T = \frac{d^2s}{dt^2} = \frac{d}{dt} \|\mathbf{r}'(t)\| \).
3. Since \( \|\mathbf{a}(t)\|^2 = a_T^2 + a_N^2 \) then find \( a_N = \sqrt{\|\mathbf{a}(t)\|^2 - a_T^2} \).
Example (2 of 2)

\[ \mathbf{r}(t) = \langle \cos t, \sin t, \sin 3t \rangle \]
Example (2 of 2)

\[ \mathbf{r}(t) = \langle \cos t, \sin t, \sin 3t \rangle \]

Steps:

\[ \frac{ds}{dt} = \sqrt{\frac{11}{2} + \frac{9}{2} \cos 6t} \]
Example (2 of 2)

\[ \mathbf{r}(t) = \langle \cos t, \sin t, \sin 3t \rangle \]

Steps:

\[ \frac{ds}{dt} = \sqrt{\frac{11}{2} + \frac{9}{2} \cos 6t} \]

\[ a_T = \frac{-27 \sin 6t}{2\sqrt{\frac{11}{2} + \frac{9}{2} \cos 6t}}. \]
Example (2 of 2)

\[ r(t) = \langle \cos t, \sin t, \sin 3t \rangle \]

Steps:

\[ \frac{ds}{dt} = \sqrt{\frac{11}{2} + \frac{9}{2} \cos 6t} \]

\[ a_T = \frac{-27 \sin 6t}{2 \sqrt{\frac{11}{2} + \frac{9}{2} \cos 6t}}. \]

\[ a_N = 2 \sqrt{\frac{23 - 18 \cos 6t}{11 + 9 \cos 6t}}. \]
Illustration

Tangential components of acceleration are shown in green.
Normal components of acceleration are shown in red.
Application: Finding Curvature

\[
a(t) = \frac{d^2 s}{dt^2} T(t) + \kappa \left( \frac{ds}{dt} \right)^2 N(t)
\]

\[
a(t) \times T(t) = \frac{d^2 s}{dt^2} T(t) \times T(t) + \kappa \left( \frac{ds}{dt} \right)^2 N(t) \times T(t) = 0
\]

\[
\| a(t) \times T(t) \| = \kappa \left( \frac{ds}{dt} \right)^2
\]

\[
\left\| \frac{r''(t) \times r'(t)}{\| r'(t) \|} \right\| = \kappa \| r'(t) \|^2
\]

\[
\frac{\| r''(t) \times r'(t) \|}{\| r'(t) \|^3} = \kappa
\]
Application: Finding the Binormal Vector

\[ a(t) = \frac{d^2s}{dt^2} T(t) + \kappa \left( \frac{ds}{dt} \right)^2 N(t) \]

\[ T(t) \times a(t) = \kappa \left( \frac{ds}{dt} \right)^2 B(t) \]

\[ \frac{r'(t) \times r''(t)}{\| r'(t) \|} = \frac{\| r'(t) \times r''(t) \|}{\| r'(t) \|^{3}} \| r'(t) \|^2 B(t) \]

\[ \frac{r'(t) \times r''(t)}{\| r'(t) \times r''(t) \|} = B(t) \]
Kepler’s Laws of Planetary Motion

1. Each planet follows an elliptical orbit, with the sun at one focus.
2. The line segment joining the sun to a planet sweeps out equal areas in equal times.
3. If $T$ is the time required for a given planet to make one orbit of the sun and if the length of the major axes of its elliptical orbit is $2a$, then $T^2 = ka^3$ for some constant $k$. 
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We can derive Kepler’s three laws from two of Newton’s Laws of Motion.
Assumptions

1. The sun is at the origin and the planet is at the terminal point of vector \( \mathbf{r}(t) \).
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1. The sun is at the origin and the planet is at the terminal point of vector $\mathbf{r}(t)$.
2. The gravitational force of attraction has a magnitude which is proportional to the inverse of the square of the distance separating the sun and the planet.
Assumptions

1. The sun is at the origin and the planet is at the terminal point of vector $\mathbf{r}(t)$.

2. The gravitational force of attraction has a magnitude which is proportional to the inverse of the square of the distance separating the sun and the planet.

\[
F(t) = ma(t) \quad \text{(Newton's 2nd Law)}
\]

\[
= - \frac{GmM}{\| \mathbf{r}(t) \|^2} \mathbf{r}(t)
\]
Kepler’s Laws (continued)

Let $r = \| \mathbf{r}(t) \|$ and note that $\frac{\mathbf{r}(t)}{\| \mathbf{r}(t) \|}$ is a unit vector, call it $\mathbf{u}$.

$$\mathbf{F} = m\mathbf{a} = -\frac{GmM}{r^2} \mathbf{u}$$

which implies
Kepler’s Laws (continued)

Let $r = \|\mathbf{r}(t)\|$ and note that $\frac{\mathbf{r}(t)}{\|\mathbf{r}(t)\|}$ is a unit vector, call it $\mathbf{u}$.

$$\mathbf{F} = m\mathbf{a} = -\frac{GmM}{r^2}\mathbf{u}$$

which implies

$$\mathbf{a} = -\frac{GM}{r^2}\mathbf{u}.$$

**Note:** this says the acceleration is always in the opposite direction of $\mathbf{r}(t)$, toward the sun.
Consider

\[
\frac{d}{dt}(\mathbf{r}(t) \times \mathbf{v}(t)) = \mathbf{r}'(t) \times \mathbf{v}(t) + \mathbf{r}(t) \times \mathbf{v}'(t)
\]

\[
= \mathbf{v}(t) \times \mathbf{v}(t) + \mathbf{r}(t) \times \mathbf{a}(t)
\]

\[
= 0 + 0 = 0
\]
Kepler’s Laws (continued)

Consider

\[ \frac{d}{dt}(r(t) \times v(t)) = r'(t) \times v(t) + r(t) \times v'(t) \]

\[ = v(t) \times v(t) + r(t) \times a(t) \]

\[ = 0 + 0 = 0 \]

Hence \( r \times v = c \) a constant vector. This implies the orbit of the planet must lie in plane.
Kepler’s First Law

1. Each planet follows an elliptical orbit, with the sun at one focus.
Kepler’s First Law

1. Each planet follows an elliptical orbit, with the sun at one focus.

Assume the planet’s orbit is in the $xy$-plane so that $\mathbf{c}$ is parallel to the $z$-axis.
1. Each planet follows an elliptical orbit, with the sun at one focus.

Assume the planet’s orbit is in the $xy$-plane so that $\mathbf{c}$ is parallel to the $z$-axis.

Since $\mathbf{r}(t) = r \mathbf{u}$, then $\mathbf{v}(t) = \mathbf{r}'(t) = \frac{d}{dt}(r \mathbf{u}) = \frac{dr}{dt} \mathbf{u} + r \frac{d\mathbf{u}}{dt}$
Kepler’s First Law (continued)

\[
c = r \times v
\]

\[
= r \times \left( \frac{dr}{dt} u + r \frac{du}{dt} \right)
\]

\[
= r \times \frac{dr}{dt} u + r \times r \frac{du}{dt}
\]

\[
= \frac{dr}{dt} r \times u + r^2 \left( u \times \frac{du}{dt} \right)
\]

\[
= r^2 \left( u \times \frac{du}{dt} \right)
\]
Kepler’s First Law (continued)

\[ \mathbf{a} \times \mathbf{c} = \mathbf{a} \times r^2 \left( \mathbf{u} \times \frac{d\mathbf{u}}{dt} \right) \]

\[ = -\frac{GM}{r^2} \mathbf{u} \times r^2 \left( \mathbf{u} \times \frac{d\mathbf{u}}{dt} \right) \]

\[ = -GM \mathbf{u} \times \left( \mathbf{u} \times \frac{d\mathbf{u}}{dt} \right) \]

\[ = -GM \left[ \left( \mathbf{u} \cdot \frac{d\mathbf{u}}{dt} \right) \mathbf{u} - (\mathbf{u} \cdot \mathbf{u}) \frac{d\mathbf{u}}{dt} \right] \]

\[ = GM \frac{d\mathbf{u}}{dt} \]
Kepler’s First Law (continued)

\[
GM \frac{du}{dt} = a \times c
\]

\[
= \frac{dv}{dt} \times c
\]

\[
= \frac{d}{dt}(v \times c) \text{ (since } c \text{ is constant)}
\]

\[
GMu + b = v \times c \text{ (where } b \text{ is constant)}
\]

**Note:** since \(v \times c\) is orthogonal to \(c\) then \(v \times c\) is a vector in the \(xy\)-plane. Vector \(u\) is in the \(xy\)-plane, so \(b\) must also be in the \(xy\)-plane. We can choose \(b\) to be parallel to \(i\).
Kepler’s First Law (continued)

\[ \|c\|^2 = c \cdot c = (r \times v) \cdot c = r \cdot (v \times c) = ru \cdot (GMu + b) = rGMu \cdot u + ru \cdot b = rGM + r\|b\| \cos \theta \]

Remark: Angle \( \theta \) is the angle between \( r \) and \( i \).
Kepler’s First Law (continued)

\[ \| \mathbf{c} \|^2 = \mathbf{c} \cdot \mathbf{c} = (\mathbf{r} \times \mathbf{v}) \cdot \mathbf{c} = \mathbf{r} \cdot (\mathbf{v} \times \mathbf{c}) = \mathbf{r} \mathbf{u} \cdot (G\mathbf{M}\mathbf{u} + \mathbf{b}) = rGM \mathbf{u} \cdot \mathbf{u} + r\mathbf{u} \cdot \mathbf{b} = 1 = rGM + r\|\mathbf{b}\| \cos \theta \]

Remark: Angle \( \theta \) is the angle between \( \mathbf{r} \) and \( \mathbf{i} \).
Kepler’s First Law (continued)

Letting $c = \|c\|$ and $b = \|b\|$ we may solve for $r$:

$$r = \frac{c^2}{GM + b \cos \theta}$$

$$= \frac{ed}{1 + e \cos \theta}$$

where $e = \frac{b}{GM}$ and $d = \frac{c^2}{b}$. 
Kepler’s First Law (continued)

Letting \( c = \|\mathbf{c}\| \) and \( b = \|\mathbf{b}\| \) we may solve for \( r \):

\[
    r = \frac{c^2}{GM + b \cos \theta} = \frac{ed}{1 + e \cos \theta}
\]

where \( e = \frac{b}{GM} \) and \( d = \frac{c^2}{b} \).

This is the equation of an ellipse in polar coordinates!
Homework

- Read Section 11.5.
- Exercises: 1–29 odd, 39, 40