Vector Fields
MATH 311, *Calculus III*

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Background

We have already discussed **vector-valued functions** which are functions of the form \( \mathbf{r} : \mathbb{R} \to V_2 \) (or \( \mathbf{r} : \mathbb{R} \to V_3 \)). They are useful for describing a path in 2- or 3-dimensional space.

**Vector field functions** are functions of the form \( \mathbf{F} : \mathbb{R}^2 \to V_2 \) (or \( \mathbf{F} : \mathbb{R}^3 \to V_3 \)) which are useful for describing flows of fluids in 2- or 3-dimensional space.
A **vector field** in the plane is a function $F(x, y)$ mapping points in $\mathbb{R}^2$ into the set of two-dimensional vectors $V_2$. We write

$$F(x, y) = \langle f_1(x, y), f_2(x, y) \rangle = f_1(x, y)i + f_2(x, y)j,$$

for scalar functions $f_1(x, y)$ and $f_2(x, y)$. In space, a **vector field** is a function $F(x, y, z)$ mapping points in $\mathbb{R}^3$ into the set of three-dimensional vectors $V_3$. We write

$$F(x, y, z) = \langle f_1(x, y, z), f_2(x, y, z), f_3(x, y, z) \rangle = f_1(x, y, z)i + f_2(x, y, z)j + f_3(x, y, z)k,$$

for scalar functions $f_1(x, y, z)$, $f_2(x, y, z)$, and $f_3(x, y, z)$. 

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for scalar functions $f_1(x, y, z)$, $f_2(x, y, z)$, and $f_3(x, y, z)$.
Consider the vector field function $\mathbf{F}(x, y) = \langle -y, x \rangle$ and plot $\mathbf{F}(1, 0), \mathbf{F}(0, 1),$ and $\mathbf{F}(1, 1)$. 
Graph of a Vector Field

The **graph of a vector field** $\mathbf{F}(x, y)$ is a two-dimensional graph with vectors $\mathbf{F}(x, y)$ plotted with initial points at $(x, y)$. 
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**Example**
Consider the graph of the vector field function $F(x, y) = \langle -y, x \rangle$. 
Example

\[ \mathbf{F}(x, y) = \langle -y, x \rangle \]
Example

\[ F(x, y) = \langle \sin x, \cos y \rangle \]
Matching Exercise

Match the following functions to the graphs of the vector fields.

\[ F(x, y) = \langle y, x \rangle \]
\[ G(x, y) = \langle 2x - 3y, 2x + 3y \rangle \]
\[ H(x, y) = \langle \sin x, \sin y \rangle \]
\[ K(x, y) = \langle \ln(1 + x^2 + y^2), x \rangle \]
Vector Fields in 3 Dimensions

\[ \mathbf{F}(x, y, z) = \langle 0, 0, z \rangle \]
We may think of the vector field $\mathbf{F}(x, y) = \langle f_1(x, y), f_2(x, y) \rangle$ as indicating the direction of movement of an object at each point in space.

If the object is initially at $(x_0, y_0)$ its future position is given by the solution to the differential equation:

$$
x'(t) = f_1(x(t), y(t))
$$
$$
y'(t) = f_2(x(t), y(t))
$$

$$(x(0), y(0)) = (x_0, y_0)$$

The parametric plot of the solution generates a flow line through $(x_0, y_0)$. 

Flow Lines
Example

Consider the vector field $\mathbf{F}(x, y) = xi - yj$. Sketch the vector field and several flow lines with different initial conditions.
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More Flow Lines
Example

Consider the vector field $\mathbf{F}(x, y) = \langle e^{-x}, 2x \rangle$. Sketch the vector field and several flow lines with different initial conditions.
Gradient and Potential

Definition
For any scalar function $f$, the vector field $\mathbf{F} = \nabla f$ is called the gradient field of function $f$. We call $f$ a potential function for $\mathbf{F}$. Whenever $\mathbf{F} = \nabla f$, for some scalar function $f$, we refer to $\mathbf{F}$ as a conservative vector field.
Example (1 of 4)

Find the gradient field of \( f(x, y, z) = x \cos \left( \frac{y}{z} \right) \).
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Find the gradient field of \( f(x, y, z) = x \cos \left( \frac{y}{z} \right) \).

\[ \nabla f(x, y, z) = \left\langle \cos \left( \frac{y}{z} \right), -\frac{x}{z} \sin \left( \frac{y}{z} \right), \frac{xy}{z^2} \sin \left( \frac{y}{z} \right) \right\rangle \]
Example (2 of 4)

Find the gradient field of

\[
f(x, y, z) = \frac{m M G}{\sqrt{x^2 + y^2 + z^2}}
\]

Remark: this is the gravitational vector field between two bodies of mass \(m\) and \(M\) whose centers of mass are separated by a distance \(\sqrt{x^2 + y^2 + z^2}\).
Example (2 of 4)

Find the gradient field of

\[ f(x, y, z) = \frac{m M G}{\sqrt{x^2 + y^2 + z^2}} \]

\[ \nabla f(x, y, z) = -\frac{m M G}{(x^2 + y^2 + z^2)^{3/2}} \langle x, y, z \rangle \]

Remark: this is the gravitational vector field between two bodies of mass \( m \) and \( M \) whose centers of mass are separated by a distance \( \sqrt{x^2 + y^2 + z^2} \).
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\]

\[
=\frac{m M G}{\left(\sqrt{x^2 + y^2 + z^2}\right)^2} \frac{-\langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}}
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**Remark**: this is the gravitational vector field between two bodies of mass \( m \) and \( M \) whose centers of mass are separated by a distance \( \sqrt{x^2 + y^2 + z^2} \).
Example (3 of 4)

Find a potential for the vector field

\[ \mathbf{F}(x, y) = \langle 2x \cos y - y \cos x, -x^2 \sin y - \sin x \rangle \]
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Find a potential for the vector field

\[ \mathbf{F}(x, y) = \langle 2x \cos y - y \cos x, -x^2 \sin y - \sin x \rangle \]

If \( \nabla f(x, y) = \mathbf{F}(x, y) \) then

\[ \frac{\partial f}{\partial x} = 2x \cos y - y \cos x \]

\[ f(x, y) = x^2 \cos y - y \sin x + h(y) \]

where \( h(y) \) is an arbitrary function of \( y \) only.
Example (3 of 4)

Find a potential for the vector field

$$\mathbf{F}(x, y) = \langle 2x \cos y - y \cos x, -x^2 \sin y - \sin x \rangle$$

If $\nabla f(x, y) = \mathbf{F}(x, y)$ then

$$\frac{\partial f}{\partial x} = 2x \cos y - y \cos x$$
$$f(x, y) = x^2 \cos y - y \sin x + h(y)$$

where $h(y)$ is an arbitrary function of $y$ only.

$$\frac{\partial f}{\partial y} = -x^2 \sin y - \sin x = -x^2 \sin y - \sin x + h'(y)$$
$$h'(y) = 0$$

Thus $h(y) = C$, a constant and

$$f(x, y) = x^2 \cos y - y \sin x + C.$$
Find a potential for the vector field

\[ \mathbf{F}(x, y) = \langle ye^x + \sin y, e^x + x \cos y + y^2 \rangle \]
Example (4 of 4)

Find a potential for the vector field

\[ \mathbf{F}(x, y) = \langle ye^x + \sin y, e^x + x \cos y + y^2 \rangle \]

\[ f(x, y) = ye^x + x \sin y + \frac{1}{3} y^3 + C \]
Homework

- Read Section 14.1.
- Exercises: 1, 5, 9, 11, 13, 15, 19–39 odd, 47–59 odd