This test consists of two halves, an in class portion and a take home portion. You have 50 minutes to complete the in class portion. The take home portion will be due at the beginning of class on Thursday, April 27, 2000.

Please answer the following questions. Partial credit will be given as appropriate, do not leave any problem blank.

**In Class Questions**

1. (8 points) Let \( f(x, y) \) be an arbitrary continuous function of two variables. Use polar coordinates to set up an iterated integral for \( f(x, y) \) over the region \( R \) shown in the figure below.
2. (8 points) Use a triple iterated integral to find the volume of the tetrahedron bounded by the coordinate planes and the plane

\[2x + 3y + 6z = 12.\]

3. (1 point each) Let \( f \) be a scalar function and \( \mathbf{F} \) be a vector field depending on the same variables. State whether each of the following expressions is meaningful. For those that are meaningful, state whether the result is a scalar or a vector.

(a) \( \nabla \times f \)

(b) \( \nabla f \)

(c) \( \nabla \cdot \mathbf{F} \)
(d) $\nabla \times (\nabla f)$

(e) $\nabla \mathbf{F}$

(f) $\nabla(\nabla \cdot \mathbf{F})$

(g) $\nabla \times (\nabla \times \mathbf{F})$

(h) $\nabla \cdot (\nabla \cdot \mathbf{F})$

(i) $(\nabla f) \times (\nabla \cdot \mathbf{F})$

(j) $\nabla \cdot (\nabla \times (\nabla \mathbf{F}))$
4. (4 points each) Consider the equation below.

\[ x^2 - y^2 - z^2 = 1 \]

(a) Change the equation above to cylindrical coordinates.

(b) Change the equation above to spherical coordinates.
5. (4 points each) Consider the function

\[ \mathbf{F}(x, y, z) = x^3 \ln z \mathbf{i} + x e^{-y} \mathbf{j} - (y^2 + 2z) \mathbf{k}. \]

(a) Find the curl of the vector field above.

(b) Find the divergence of the vector field above.
6. (8 points) Show that the vector field below is conservative by finding a potential for the field.

\[ \mathbf{F}(x, y, z) = 2x \sin z \mathbf{i} + 2y \cos z \mathbf{j} + (x^2 \cos z - y^2 \sin z) \mathbf{k} \]
Take Home Questions

1. (8 points) Find the volume of the solid region bounded between the paraboloids \( z = x^2 + y^2 \) and \( z = 36 - 3x^2 - 3y^2 \).

2. (8 points) Consider a solid region bounded between the sphere centered at the origin with radius \( R_1 > 0 \) and the sphere centered at the origin with radius \( R_2 > R_1 \). If the density of the material making up the region at point \((x, y, z)\) is inversely proportional to the distance of the point from the origin, find the mass of the solid.

3. (10 points) Consider the vector field

\[
\mathbf{F}(x, y, z) = (3x^2yz - 3y)i + (x^3z - 3x)j + (x^3y + 2z)k.
\]

Evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \), where \( C \) is the curve with initial point \((0, 0, 2)\) and terminal point \((0, 3, 0)\) shown in the figure below.

![Diagram](image)

4. (8 points) Evaluate the following integral by changing to spherical coordinates.

\[
\int_0^3 \int_0^{\sqrt{9-y^2}} \int_0^{\sqrt{18-x^2-y^2}} (x^2 + y^2 + z^2) \, dz \, dx \, dy
\]
5. (8 points) Suppose that as a particle moves through a conservative vector field, its potential energy is decreasing at a rate of $k$ units per second. At what rate is its kinetic energy changing?

6. (8 points) In addition to the gradient, the curl, and the divergence there is another operation which can be performed on a vector field. It is called the Laplacian. The Laplacian $\nabla^2$, is the divergence of the gradient, in other words,

$$\nabla^2 f = \nabla \cdot (\nabla f) = \nabla \cdot \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2},$$

where $f$ is a scalar function. The Laplacian can also be applied to a vector function $\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ by applying it component-wise, in other words

$$\nabla^2 \mathbf{F} = \nabla^2 M\mathbf{i} + \nabla^2 N\mathbf{j} + \nabla^2 P\mathbf{k}.$$

Use the definition of the Laplacian to prove the following identity assuming that all appropriate derivatives exist and are continuous.

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla (\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$$