Coordinates and Basis
MATH 322, *Linear Algebra I*

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Today we will explore the concepts of

- generalized coordinate systems, and
- the dimension of a (finite dimensional) vector space.
Nonrectangular Coordinate Systems

In brief, any linearly independent set of vectors can be used as coordinate axes.

Vectors which define a coordinate system are called **basis vectors**. In $V = \mathbb{R}^3$ we are familiar with $\{i, j, k\}$. 
Remark: a vector space $V$ is said to be **finite-dimensional** if there is a finite set of vectors in $V$ which spans $V$. If there is no such finite set, $V$ is said to be **infinite-dimensional**.

**Definition**
If $V$ is a vector space and $S = \{v_1, v_2, \ldots, v_n\}$ is a set of vectors in $V$, then $S$ is a **basis** for $V$ provided,

1. $S$ is linearly independent, and
2. $V = \text{span}(S)$. 
Examples (1 of 3)

- Standard basis for $\mathbb{R}^n$ is

$$
e_1 = (1, 0, 0, \ldots, 0)$$
$$e_2 = (0, 1, 0, \ldots, 0)$$
$$\vdots$$
$$e_n = (0, 0, 0, \ldots, 1).$$
Examples (1 of 3)

- Standard basis for $\mathbb{R}^n$ is

\[
\begin{align*}
e_1 &= (1, 0, 0, \ldots, 0) \\
e_2 &= (0, 1, 0, \ldots, 0) \\
\vdots \\
e_n &= (0, 0, 0, \ldots, 1).
\end{align*}
\]

- Standard basis for $P_n$ is

\[
\begin{align*}
p_0 &= 1 \\
p_1 &= x \\
p_2 &= x^2 \\
\vdots \\
p_n &= x^n.
\end{align*}
\]
Examples (2 of 3)

- Standard basis for $\mathbb{R}^3$ is

$$
\begin{align*}
  e_1 &= i = (1, 0, 0) \\
  e_2 &= j = (0, 1, 0) \\
  e_3 &= k = (0, 0, 1).
\end{align*}
$$
Examples (2 of 3)

- Standard basis for $\mathbb{R}^3$ is

$$
e_1 = i = (1, 0, 0)$$
$$e_2 = j = (0, 1, 0)$$
$$e_3 = k = (0, 0, 1).$$

- Another basis for $\mathbb{R}^3$ is

$$v_1 = (1, 2, 1)$$
$$v_2 = (2, 9, 0)$$
$$v_3 = (3, 3, 4).$$
Examples (3 of 3)

- Standard basis for $M_{22}$ (the vector space of $2 \times 2$ matrices) is

\[
M_1 = \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix},
M_2 = \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix},
M_3 = \begin{bmatrix}
0 & 0 \\
1 & 0
\end{bmatrix},
M_4 = \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}.
\]
Examples (3 of 3)

- Standard basis for $M_{22}$ (the vector space of $2 \times 2$ matrices) is

  $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

  $M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

  $M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

  $M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

- The vector space $P_{\infty}$ is infinite-dimensional. Why?
Basis Representation

**Theorem**

*If* $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *is a basis for* $V$, *then every* $\mathbf{v} \in V$ *can be expressed as*

$$\mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_n \mathbf{v}_n$$

*where the* $c_i$’s *are unique.*
Proof

Since $S$ spans $V$ then any $v \in V$ can be expressed as a linear combination of the elements of $S$.

$$v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$$
Proof

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$$v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$$

Suppose there are other coefficients $k_1, k_2, \ldots, k_n$ for which

$$v = k_1v_1 + k_2v_2 + \cdots + k_nv_n$$
Proof

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- Suppose there are other coefficients $k_1, k_2, \ldots, k_n$ for which

$$v = k_1v_1 + k_2v_2 + \cdots + k_nv_n$$

- Subtract the vectors,

$$0 = v - v = (c_1 - k_1)v_1 + (c_2 - k_2)v_2 + \cdots + (c_n - k_n)v_n.$$
Proof

- Since $S$ spans $V$ then any $\mathbf{v} \in V$ can be expressed as a linear combination of the elements of $S$.

\[ \mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_n \mathbf{v}_n \]

- Suppose there are other coefficients $k_1, k_2, \ldots, k_n$ for which

\[ \mathbf{v} = k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2 + \cdots + k_n \mathbf{v}_n \]

- Subtract the vectors,

\[ 0 = \mathbf{v} - \mathbf{v} = (c_1 - k_1) \mathbf{v}_1 + (c_2 - k_2) \mathbf{v}_2 + \cdots + (c_n - k_n) \mathbf{v}_n. \]

- Since $S$ is a linearly independent set $c_i = k_i$ for $i = 1, 2, \ldots, n$. 
Definition
If \( S = \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \} \) is a basis for \( V \) and \( \mathbf{v} \in V \) is written as

\[
\mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_n \mathbf{v}_n
\]

then the scalars \( c_1, c_2, \ldots, c_n \) are called the coordinates of \( \mathbf{v} \) relative to the basis \( S \). The vector \( (c_1, c_2, \ldots, c_n) \in \mathbb{R}^n \) is called the coordinate vector of \( \mathbf{v} \) relative to \( S \).

Notation: \((\mathbf{v})_S = (c_1, c_2, \ldots, c_n)\)
Definition
If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for $V$ and $\mathbf{v} \in V$ is written as

$$\mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_n \mathbf{v}_n$$

then the scalars $c_1, c_2, \ldots, c_n$ are called the coordinates of $\mathbf{v}$ relative to the basis $S$. The vector $(c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$ is called the coordinate vector of $\mathbf{v}$ relative to $S$.

Notation: $(\mathbf{v})_S = (c_1, c_2, \ldots, c_n)$

Remark: $(c_1, c_2, \ldots, c_n)$ is an ordered $n$-tuple and the order is significant.
Notation

- The notation,
  \[(v)_S = (c_1, c_2, \ldots, c_n)\]
will be called the **comma-delimited form** of the coordinate vector.

- Sometimes it will be more convenient to write the coordinate vector as a column matrix or row matrix. For this we will use the notations

\[
[v]_S = \begin{bmatrix}
  c_1 \\
  c_2 \\
  \vdots \\
  c_n
\end{bmatrix}
\text{ or }
[v]_S = \begin{bmatrix}
  c_1 \\
  \vdots \\
  c_n
\end{bmatrix}.
\]
Correspondence Between $\mathbb{R}^n$ and $V$
Examples

- Find the coordinate vector of \((1, -2, 5)\) in the standard basis for \(\mathbb{R}^3\).
- Find the coordinate vector of \((a, b, c)\) in the standard basis for \(\mathbb{R}^3\).
- Find the coordinate vector of \((1, -2, 5)\) in the basis \(\{(1, 2, 1), (2, 9, 0), (3, 3, 4)\}\) for \(\mathbb{R}^3\).
- Find the coordinate vector for the polynomial

\[ p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \]

in the standard basis for \(P_n\).
Homework

- Read Section 4.4
- Exercises: 1–7 odd, 8, 10, 11, 19, 21, 22, 25.