Properties of the Determinant Function

MATH 322, *Linear Algebra I*

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Spring 2015
Overview

Today’s discussion will illuminate some of the properties of the determinant:

▶ relationship with scalar products
▶ multi-linearity
▶ relationship with matrix products
▶ relationship with invertible matrices
Basic Properties

Theorem

*If $A$ is an $n \times n$ matrix and $k$ is a scalar, then* \[ \det(kA) = k^n \det(A). \]
Basic Properties

Theorem
If $A$ is an $n \times n$ matrix and $k$ is a scalar, then
$\det(kA) = k^n \det(A)$.

Proof.

\[
\begin{vmatrix}
ka_{11} & ka_{12} & \cdots & ka_{1n} \\
ka_{21} & ka_{22} & \cdots & ka_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
ka_{n1} & ka_{n2} & \cdots & ka_{nn}
\end{vmatrix} = k^n
\begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix}
\]
A Non-property

We can show by example that det($A + B$) $\neq$ det($A$) + det($B$).

Example

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then

$$
\det(A + B) = \begin{vmatrix} 2 & 2 \\ 3 & 5 \end{vmatrix} = 4 \neq 1 + (-2) = \det(A) + \det(B).
$$
However . . .

If matrix $A$ and $B$ differ in just a single row . . .

**Theorem**

Let $A$, $B$, and $C$ be $n \times n$ matrices that differ only in a single row, say the $r^{th}$ row, and suppose the $r^{th}$ of $C$ is the sum of the $r^{th}$ rows of $A$ and $B$. Then $\det(C) = \det(A) + \det(B)$. 

**Remark:** the same result holds for columns.

**Example**

Let

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

and

$B = \begin{bmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{bmatrix}$,

and

$C = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$,

compare $\det(A) + \det(B)$ to $\det(C)$. 
However . . .

If matrix $A$ and $B$ differ in just a single row . . .

**Theorem**

Let $A$, $B$, and $C$ be $n \times n$ matrices that differ only in a single row, say the $r^{th}$ row, and suppose the $r^{th}$ of $C$ is the sum of the $r^{th}$ rows of $A$ and $B$. Then $\text{det}(C) = \text{det}(A) + \text{det}(B)$.

**Remark:** the same result holds for columns.
However . . .

If matrix $A$ and $B$ differ in just a single row . . .

**Theorem**

Let $A$, $B$, and $C$ be $n \times n$ matrices that differ only in a single row, say the $r^{th}$ row, and suppose the $r^{th}$ row of $C$ is the sum of the $r^{th}$ rows of $A$ and $B$. Then $\det(C) = \det(A) + \det(B)$.

**Remark:** the same result holds for columns.

**Example**

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{bmatrix}$, and $C = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$, compare $\det(A) + \det(B)$ to $\det(C)$. 
Theorem

If $A$ and $B$ are $n \times n$ matrices then $\det(AB) = \det(A) \det(B)$. 
Determinant of Matrix Product

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Remark: we will first prove this for the case where $A$ is an elementary matrix.
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If $A$ and $B$ are $n \times n$ matrices then $\det(AB) = \det(A) \det(B)$.

Remark: we will first prove this for the case where $A$ is an elementary matrix.

Lemma
Suppose $E$ is an elementary $n \times n$ matrix and $B$ is an $n \times n$ matrix, then $\det(EB) = \det(E) \det(B)$. 
Determinant of Matrix Product

**Theorem**

*If A and B are n × n matrices then det(AB) = det(A) det(B).*

**Remark:** we will first prove this for the case where A is an elementary matrix.

**Lemma**

*Suppose E is an elementary n × n matrix and B is an n × n matrix, then det(EB) = det(E) det(B).*
Determinant of Matrix Product

Theorem

*If* $A$ and $B$ are $n \times n$ matrices then $\det(AB) = \det(A) \det(B)$.

**Remark:** we will first prove this for the case where $A$ is an elementary matrix.

Lemma

*Suppose* $E$ is an elementary $n \times n$ matrix and $B$ is an $n \times n$ matrix, *then* $\det(EB) = \det(E) \det(B)$.

If $B$ is an $n \times n$ matrix and $E_1, E_2, \ldots, E_r$ are elementary matrices then

$$\det(E_1 E_2 \cdots E_mB) = \det(E_1) \det(E_2) \cdots \det(E_m) \det(B)$$
Proof (for first elementary row operation)

- Suppose $E$ results from multiplying the $j$th row of $I_n$ by scalar $k$.

\[ \det(E) = k \]
Proof (for first elementary row operation)

- Suppose $E$ results from multiplying the $j$th row of $I_n$ by scalar $k$.
  \[
  \det(E) = k
  \]
- Matrix product $EB$ multiplies the $j$th row of $B$ by $k$.
  \[
  \det(EB) = k \det(B)
  \]
Proof (for first elementary row operation)

- Suppose $E$ results from multiplying the $j$th row of $I_n$ by scalar $k$.
  \[ \det(E) = k \]

- Matrix product $EB$ multiplies the $j$th row of $B$ by $k$.
  \[ \det(EB) = k \det(B) = \det(E) \det(B) \]
**Theorem**

A square matrix $A$ is invertible if and only if $\det(A) \neq 0$. 

**Remark:** this is one of the most important results of linear algebra.
Invertibility

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**Remark:** This is one of the most important results of linear algebra.
Proof

Let $E_1, E_2, \ldots, E_r$ be the elementary matrices which place $A$ in reduced row echelon form, $R$.

$$R = E_r \cdots E_2 E_1 A$$

$$\det(R) = \det(E_r) \cdots \det(E_2) \det(E_1) \det(A)$$
Proof

Let $E_1, E_2, \ldots, E_r$ be the elementary matrices which place $A$ in reduced row echelon form, $R$.

$$R = E_r \cdots E_2 E_1 A$$
$$\det(R) = \det(E_r) \cdots \det(E_2) \det(E_1) \det(A)$$

Since the determinant of an elementary matrix is nonzero then $\det(A)$ and $\det(R)$ are either both zero or both nonzero.
Proof

Let $E_1, E_2, \ldots, E_r$ be the elementary matrices which place $A$ in reduced row echelon form, $R$.

\[
R = E_r \cdots E_2 E_1 A \\
\det(R) = \det(E_r) \cdots \det(E_2) \det(E_1) \det(A)
\]

Since the determinant of an elementary matrix is nonzero then $\det(A)$ and $\det(R)$ are either both zero or both nonzero.

If $A$ is invertible, the reduced row echelon form of $A$ is $I$. In this case $\det(R) = \det(I) = 1$ which implies $\det(A) \neq 0$. 
Proof

- Let $E_1, E_2, \ldots, E_r$ be the elementary matrices which place $A$ in reduced row echelon form, $R$.

$$
R = E_r \cdots E_2 E_1 A \\
\det(R) = \det(E_r) \cdots \det(E_2) \det(E_1) \det(A)
$$

- Since the determinant of an elementary matrix is nonzero then $\det(A)$ and $\det(R)$ are either both zero or both nonzero.

- If $A$ is invertible, the reduced row echelon form of $A$ is $I$. In this case $\det(R) = \det(I) = 1$ which implies $\det(A) \neq 0$.

- If $\det(A) \neq 0$, then $\det(R) \neq 0$ which implies $R$ cannot have a row of all zeros. This implies $R = I$ and thus $A$ is invertible.
Example

Determine if the matrix below is invertible.

\[
\begin{bmatrix}
  2 & 6 & 2 \\
  7 & 6 & 3 \\
  7 & 2 & 7 \\
\end{bmatrix}
\]
We can now prove the general case of the theorem:

**Theorem**

*If A and B are \( n \times n \) matrices then \( \det(AB) = \det(A) \det(B) \).*
We can now prove the general case of the theorem:

**Theorem**

*If A and B are n × n matrices then det(AB) = det(A) det(B).*

**Proof.**

Case: A is singular.

- AB is not invertible.
- By last theorem det(AB) = 0 = (0) det(B) = det(A) det(B).
Proof (continued)

Case: $A$ is invertible.

- $A$ can be expressed as a product of elementary matrices.

$$ A = E_1 E_2 \cdots E_r $$

Applying the last theorem yields

$$ \det(AB) = \det(E_1 E_2 \cdots E_r) \det(B) = \det(A) \det(B). $$
Case: $A$ is invertible.

- $A$ can be expressed as a product of elementary matrices.

\[ A = E_1 E_2 \cdots E_r \]

\[ AB = E_1 E_2 \cdots E_r B \]
Proof (continued)

Case: $A$ is invertible.

- $A$ can be expressed as a product of elementary matrices.

$$A = E_1 E_2 \cdots E_r$$
$$AB = E_1 E_2 \cdots E_r B$$

- Applying the last theorem yields

$$\det(AB) = \det(E_1) \det(E_2) \cdots \det(E_r) \det(B)$$
$$= \det(E_1 E_2 \cdots E_r) \det(B)$$
$$= \det(A) \det(B).$$
Example

Consider the matrices

\[ A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \]
\[ B = \begin{bmatrix} -1 & 3 \\ 5 & 8 \end{bmatrix} \]
\[ AB = \begin{bmatrix} 2 & 17 \\ 3 & 14 \end{bmatrix} \]

and verify that \( \det(AB) = \det(A) \det(B) \).
A Corollary

Corollary

*If A is invertible, then*

\[
\det(A^{-1}) = \frac{1}{\det(A)} = (\det(A))^{-1}.
\]
A Corollary

**Corollary**

*If $A$ is invertible, then*

\[
\det(A^{-1}) = \frac{1}{\det(A)} = (\det(A))^{-1}.
\]

**Proof.**

- $A^{-1}A = I$ and $\det(A^{-1}A) = \det(I) = 1$.
- Since $\det(A^{-1}A) = \det(A^{-1})\det(A) = 1$ and $\det(A) \neq 0$, then

\[
\det(A^{-1}) = \frac{1}{\det(A)}.
\]
Observations:

- If we multiply the entries of a row or column of a square matrix by their corresponding cofactors and add the results, we obtain the determinant of the matrix.

Example

Let

\[
\begin{pmatrix}
1 & 3 & 2 \\
4 & 6 & 7 \\
5 & 0 & 1 \\
\end{pmatrix}
\]

and find the following sum.

\[a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23}\]
Observations:

- If we multiply the entries of a row or column of a square matrix by their corresponding cofactors and add the results, we obtain the determinant of the matrix.

- If we multiply the entries in a row (column) of a matrix by the corresponding cofactors from a different row (column), the sum of these products will be 0.

Example

Let \( A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 6 & 7 \\ 5 & 0 & 1 \end{bmatrix} \) and find the following sum.

\[
a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23}
\]
Definition
If $A$ is an $n \times n$ matrix and $C_{ij}$ is the cofactor of $a_{ij}$, then the matrix

$$
\begin{bmatrix}
C_{11} & C_{12} & \cdots & C_{1n} \\
C_{21} & C_{22} & \cdots & C_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
C_{n1} & C_{n2} & \cdots & C_{nn}
\end{bmatrix}
$$

is called the **matrix of cofactors from** $A$. The transpose of this matrix is called the **adjoint of** $A$ and is denoted $\text{adj}(A)$. 
Example

Let \( A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 6 & 7 \\ 5 & 0 & 1 \end{bmatrix} \) and find the matrix of cofactors and \( \text{adj}(A) \).
Example

Let \( A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 6 & 7 \\ 5 & 0 & 1 \end{bmatrix} \) and find the matrix of cofactors and \( \text{adj}(A) \).

Matrix of cofactors: \[
\begin{bmatrix} 6 & 31 & -30 \\ -3 & -9 & 15 \\ 9 & 1 & -6 \end{bmatrix}
\]
Example

Let \( A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 6 & 7 \\ 5 & 0 & 1 \end{bmatrix} \) and find the matrix of cofactors and \( \text{adj}(A) \).

Matrix of cofactors:
\[
\begin{bmatrix}
6 & 31 & -30 \\
-3 & -9 & 15 \\
9 & 1 & -6
\end{bmatrix}
\]

\[
\text{adj}(A) = \begin{bmatrix} 6 & -3 & 9 \\ 31 & -9 & 1 \\ -30 & 15 & -6 \end{bmatrix}
\]
Inverses and Adjoint

Theorem

If $A$ is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A).$$
Inverses and Adjoints

Theorem

If $A$ is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A).$$

Proof.

$$A \text{adj}(A) = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & & \vdots \\
a_{i1} & a_{i2} & \cdots & a_{in} \\
\vdots & \vdots & & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix} \begin{bmatrix}
C_{11} & C_{21} & \cdots & C_{j1} & \cdots & C_{n1} \\
C_{12} & C_{22} & \cdots & C_{j2} & \cdots & C_{n2} \\
\vdots & \vdots & & \vdots & & \vdots \\
C_{1n} & C_{2n} & \cdots & C_{jn} & \cdots & C_{nn}
\end{bmatrix}$$
Proof (continued)

The \( ij \)th entry of \( A \text{adj}(A) \) is

\[
a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn} = \begin{cases} 
\det(A) & \text{if } i = j \\
0 & \text{if } i \neq j.
\end{cases}
\]

Thus

\[
A \text{adj}(A) = \det(A)I
\]

\[
\frac{1}{\det(A)} A \text{adj}(A) = I
\]

\[
A \left[ \frac{1}{\det(A)} \text{adj}(A) \right] = I
\]

\[
A^{-1}A \left[ \frac{1}{\det(A)} \text{adj}(A) \right] = A^{-1}I
\]

\[
\frac{1}{\det(A)} \text{adj}(A) = A^{-1}
\]
Example

Let \( A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 6 & 7 \\ 5 & 0 & 1 \end{bmatrix} \) and use the formula \( \frac{1}{\det(A)} \adj(A) \) to find \( A^{-1} \).
Example

Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 6 & 7 \\ 5 & 0 & 1 \end{bmatrix}$ and use the formula $\frac{1}{\det(A)} \text{adj}(A)$ to find $A^{-1}$.

$\det(A) = 39$

$\text{adj}(A) = \begin{bmatrix} 6 & -3 & 9 \\ 31 & -9 & 1 \\ -30 & 15 & -6 \end{bmatrix}$

$A^{-1} = \frac{1}{39} \begin{bmatrix} 6 & -3 & 9 \\ 31 & -9 & 1 \\ -30 & 15 & -6 \end{bmatrix} = \begin{bmatrix} 2/13 & -1/13 & 3/13 \\ 31/39 & -3/13 & 1/39 \\ -10/13 & 5/13 & -2/13 \end{bmatrix}$
**Cramer’s Rule**

**Cramer’s Rule** is a theoretical method for determining the solution to a linear system. It is computationally inefficient.

**Theorem**

If \( A \mathbf{x} = \mathbf{b} \) is a system of \( n \) linear equations in \( n \) unknowns such that \( \det(A) \neq 0 \), then the system has a unique solution:

\[
\begin{align*}
    x_1 &= \frac{\det(A_1)}{\det(A)}, \\
    x_2 &= \frac{\det(A_2)}{\det(A)}, \\
    &\vdots \\
    x_n &= \frac{\det(A_n)}{\det(A)}.
\end{align*}
\]

where \( A_j \) is the matrix obtained when the entries in the \( j \)th column of \( A \) are replaced by \( \mathbf{b} \).
Cramer’s Rule is a theoretical method for determining the solution to a linear system. It is computationally inefficient.

Theorem
If $Ax = b$ is a system of $n$ linear equations in $n$ unknowns such that $\det(A) \neq 0$, then the system has a unique solution:

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \ldots, \quad x_n = \frac{\det(A_n)}{\det(A)}.$$ 

where $A_j$ is the matrix obtained when the entries in the $j^{th}$ column of $A$ are replaced by $b$. 
Proof (1 of 2)

If \( \det(A) \neq 0 \) then \( A \) is invertible and the solution is \( x = A^{-1}b \).

\[
x = \frac{1}{\det(A)} \text{adj}(A)b = \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
\]

\[
= \frac{1}{\det(A)} \begin{bmatrix} b_1 C_{11} + b_2 C_{21} + \cdots + b_n C_{n1} \\ b_1 C_{12} + b_2 C_{22} + \cdots + b_n C_{n2} \\ \vdots \\ b_1 C_{1n} + b_2 C_{2n} + \cdots + b_n C_{nn} \end{bmatrix}
\]

The \( j \)th entry of \( x \) is

\[
x_j = \frac{b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{nj}}{\det(A)}
\]
Proof (2 of 2)

Let \( A_j \) be matrix \( A \) with its \( j \)th column replaced with \( b \).

\[
A_j = \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1j-1} & b_1 & a_{1j+1} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2j-1} & b_2 & a_{2j+1} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \cdots & a_{nj-1} & b_n & a_{nj+1} & \cdots & a_{nn}
\end{bmatrix}
\]

Find the \( \det(A_j) \) by cofactor expansion along the \( j \)th column.

\[
\det(A_j) = b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{nj}
\]

Consequently \( x_j = \frac{\det(A_j)}{\det(A)} \).
Example

Use Cramer’s Rule to solve the following linear system.

\[3x_1 + 3x_2 + x_3 = 1\]
\[x_1 - 4x_3 = 1\]
\[x_1 - 3x_2 + 5x_3 = 1\]
Efficiency

**Remark:** for linear systems with more than 3 equations and 3 unknowns, Gaussian elimination is more efficient for finding the solution than Cramer’s Rule.
Theorem

If $A$ is an $n \times n$ matrix, then the following statements are equivalent:

1. $A$ is invertible.
2. $Ax = 0$ has only the trivial solution.
3. The reduced row echelon form of $A$ is $I_n$.
4. $A$ is expressible as the product of elementary matrices.
5. $Ax = b$ is consistent for every $n \times 1$ matrix $b$.
6. $Ax = b$ has exactly one solution for every $n \times 1$ matrix $b$.
7. $\det(A) \neq 0$. 

Equivalent Statements
Homework

- Read Section 2.3
- Exercises: 1, 3, 5, 7, 13, 17, 19, 25, 33