Diagonalization

MATH 322, Linear Algebra I

J. Robert Buchanan

Department of Mathematics

Spring 2015
Motivation

Today we consider two fundamental questions:

▶ Given an $n \times n$ matrix $A$, does there exist a basis for $\mathbb{R}^n$ consisting of eigenvectors of $A$?

▶ Given an $n \times n$ matrix $A$, does there exist an invertible matrix $P$ such that $P^{-1}AP$ is a diagonal matrix?
Similarity Transformations

- Suppose $A$ and $P$ are $n \times n$ matrices and suppose $P$ is invertible.
- We will call the transformation $A \rightarrow P^{-1}AP = B$ a similarity transformation.
Similarity Transformations

- Suppose $A$ and $P$ are $n \times n$ matrices and suppose $P$ is invertible.
- We will call the transformation $A \rightarrow P^{-1}AP = B$ a similarity transformation.
- Matrices $A$ and $B$ share many properties in common. For instance

\[
det(B) = det(P^{-1}AP) = det(P^{-1}) det(A) det(P) = det(A) \frac{det(P)}{det(P)} = det(A).
\]
Similarity Invariants

Any property that is preserved by a similarity transformation is called a similarity invariant.

Properties of a matrix which are invariant under similarity include:

- Determinant
- Invertibility
- Rank
- Nullity
- Trace
- Characteristic polynomial
- Eigenvalues
- Eigenspace dimension
Similar Matrices

Definition
If $A$ and $B$ are square matrices, then we say $B$ is similar to $A$ if there is an invertible matrix $P$ such that $B = P^{-1}AP$.

Remark: if $B$ is similar to $A$ then $A$ is similar to $B$ since

$$B = P^{-1}AP$$
$$PBP^{-1} = PP^{-1}APP^{-1}$$
$$(P^{-1})^{-1}BP^{-1} = A$$
$$Q^{-1}BQ = A$$

where $Q = P^{-1}$ is invertible.
Diagonalizable Matrices

Definition
A square matrix $A$ is \underline{diagonalizable} if there is an invertible matrix $P$ such that $P^{-1}AP$ is a diagonal matrix. The matrix $P$ is said to \underline{diagonalize} $A$. 
Diagonalizable Matrices

Definition
A square matrix $A$ is **diagonalizable** if there is an invertible matrix $P$ such that $P^{-1}AP$ is a diagonal matrix. The matrix $P$ is said to **diagonalize** $A$.

Theorem
*If $A$ is an $n \times n$ matrix, then the following statements are equivalent.*

1. $A$ is diagonalizable.
2. $A$ has $n$ linearly independent eigenvectors.
Proof (1 $\implies$ 2)

- Suppose $A$ is diagonalizable, then $D = P^{-1}AP$ is a diagonal matrix.

$$PD = AP$$
Proof (1 \implies 2)

- Suppose $A$ is diagonalizable, then $D = P^{-1}AP$ is a diagonal matrix.

\[ PD = AP \]

- Let the diagonal entries of $D$ be $\lambda_1, \lambda_2, \ldots, \lambda_n$ and let the columns of $P$ be the vectors $p_1, p_2, \ldots, p_n$.

\[ PD = \begin{bmatrix} \lambda_1 p_1 & \lambda_2 p_2 & \cdots & \lambda_n p_n \end{bmatrix} = \begin{bmatrix} Ap_1 & Ap_2 & \cdots & Ap_n \end{bmatrix} = AP \]
Proof (1 $\implies$ 2)

- Suppose $A$ is diagonalizable, then $D = P^{-1}AP$ is a diagonal matrix.

  \[ PD = AP \]

- Let the diagonal entries of $D$ be $\lambda_1, \lambda_2, \ldots, \lambda_n$ and let the columns of $P$ be the vectors $p_1, p_2, \ldots, p_n$.

  \[ PD = \begin{bmatrix} \lambda_1 p_1 & \lambda_2 p_2 & \cdots & \lambda_n p_n \end{bmatrix} = \begin{bmatrix} Ap_1 & Ap_2 & \cdots & Ap_n \end{bmatrix} = AP \]

- Thus $Ap_1 = \lambda_1 p_1$, $Ap_2 = \lambda_2 p_2$, $\ldots$, $Ap_n = \lambda_2 p_n$. Thus the columns of $P$ are the eigenvectors of $A$. Since $P$ is invertible, \text{rank}(P) = n$ which implies its columns (the eigenvectors of $A$) are linearly independent.
Proof ($1 \implies 2$)

- Suppose $A$ is diagonalizable, then $D = P^{-1}AP$ is a diagonal matrix.
  \[ PD = AP \]

- Let the diagonal entries of $D$ be $\lambda_1, \lambda_2, \ldots, \lambda_n$ and let the columns of $P$ be the vectors $p_1, p_2, \ldots, p_n$.
  \[ PD = \begin{bmatrix} \lambda_1 p_1 & \lambda_2 p_2 & \cdots & \lambda_n p_n \end{bmatrix} = \begin{bmatrix} Ap_1 & Ap_2 & \cdots & Ap_n \end{bmatrix} = AP \]

- Thus $Ap_1 = \lambda_1 p_1$, $Ap_2 = \lambda_2 p_2$, \ldots, $Ap_n = \lambda_2 p_n$. Thus the columns of $P$ are the eigenvectors of $A$.

- Since $P$ is invertible, $\text{rank}(P) = n$ which implies its columns (the eigenvectors of $A$) are linearly independent.
Proof ($2 \implies 1$)

- Suppose $A$ has $n$ linearly independent eigenvectors $p_1, p_2, \ldots, p_n$ with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.
- Let the eigenvectors of $A$ be the columns of matrix $P$ and let $D$ be the diagonal matrix with the eigenvalues of $A$ on the diagonal.

$$AP = \begin{bmatrix} Ap_1 & Ap_2 & \cdots & Ap_n \end{bmatrix} = \begin{bmatrix} \lambda_1 p_1 & \lambda_2 p_2 & \cdots & \lambda_n p_n \end{bmatrix} = PD$$
Proof ($2 \implies 1$)

- Suppose $A$ has $n$ linearly independent eigenvectors $p_1, p_2, \ldots, p_n$ with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

- Let the eigenvectors of $A$ be the columns of matrix $P$ and let $D$ be the diagonal matrix with the eigenvalues of $A$ on the diagonal.

  \[
  AP = \begin{bmatrix} Ap_1 & Ap_2 & \cdots & Ap_n \end{bmatrix} = \begin{bmatrix} \lambda_1 p_1 & \lambda_2 p_2 & \cdots & \lambda_n p_n \end{bmatrix} = PD
  \]

- Since the columns of $P$ are linearly independent, $P$ is invertible. Therefore $D = P^{-1}AP$.  

Theorem

- If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigenvalues of a matrix $A$ and if $v_1, v_2, \ldots, v_k$ are corresponding eigenvectors then \( \{v_1, v_2, \ldots, v_k\} \) is a linearly independent set.
- An $n \times n$ matrix with $n$ distinct eigenvalues is diagonalizable.
Diagonalizing a Matrix

Given an $n \times n$ matrix $A$,

Steps:

1. Find $n$ linearly independent eigenvectors of $A$, say $p_1, p_2, \ldots, p_n$.
2. Form the matrix $P$ having $p_1, p_2, \ldots, p_n$ as its columns.
3. The matrix $P^{-1}AP$ will be diagonal with diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$, where $\lambda_i$ is the eigenvalue corresponding to $p_i$ for $i = 1, 2, \ldots, n$. 
Example

Diagonalize the matrix below.

\[ A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix} \]
Solution (1 of 2)

\[ 0 = \det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & 1 & 2 \\ -2 & \lambda & 2 \\ -2 & 1 & \lambda + 1 \end{vmatrix} = \lambda(\lambda - 1)^2 \]

Eigenvalues: \( \lambda_1 = 0, \lambda_2 = \lambda_3 = 1. \)
Solution (1 of 2)

\[ 0 = \det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & 1 & 2 \\ -2 & \lambda & 2 \\ -2 & 1 & \lambda + 1 \end{vmatrix} = \lambda(\lambda - 1)^2 \]

Eigenvalues: \( \lambda_1 = 0, \lambda_2 = \lambda_3 = 1. \)

Eigenvectors: \( p_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ p_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \ p_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}. \)
Solution (1 of 2)

\[ 0 = \det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & 1 & 2 \\ -2 & \lambda & 2 \\ -2 & 1 & \lambda + 1 \end{vmatrix} = \lambda(\lambda - 1)^2 \]

Eigenvalues: \( \lambda_1 = 0, \lambda_2 = \lambda_3 = 1. \)

Eigenvectors: \( p_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ p_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \ p_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}. \)

\[ P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } P^{-1} = \begin{bmatrix} -2 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & -1 \end{bmatrix} \]
We can verify

\[
P^{-1} AP = \begin{bmatrix}
-2 & 1 & 2 \\
1 & 0 & -1 \\
2 & -1 & -1
\end{bmatrix}
\begin{bmatrix}
3 & -1 & -2 \\
2 & 0 & -2 \\
2 & -1 & -1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 0 \\
1 & 0 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} = D
\]
Example

Verify that the matrix below is not diagonalizable.

\[ B = \begin{bmatrix} 19 & -9 & -6 \\ 25 & -11 & -9 \\ 17 & -9 & -4 \end{bmatrix} \]
Solution

\[ 0 = \det(\lambda I - B) = \left| \begin{array}{ccc} \lambda - 19 & 9 & 6 \\ -25 & \lambda + 11 & 9 \\ -17 & 9 & \lambda + 4 \end{array} \right| = (\lambda - 2)(\lambda - 1)^2 \]

Eigenvalues: \( \lambda_1 = 2, \lambda_2 = \lambda_3 = 1. \)
Solution

$$0 = \det(\lambda I - B) = \begin{vmatrix} \lambda - 19 & 9 & 6 \\ -25 & \lambda + 11 & 9 \\ -17 & 9 & \lambda + 4 \end{vmatrix} = (\lambda - 2)(\lambda - 1)^2$$

Eigenvalues: $\lambda_1 = 2$, $\lambda_2 = \lambda_3 = 1$.

Eigenvectors: $p_1 = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$, $p_2 = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$.

Since there are only 2 linearly independent eigenvectors for this matrix, the matrix is not diagonalizable.
Example

Verify the following matrix is diagonalizable.

\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
0 & 5 & 6 & 7 \\
0 & 0 & 8 & 9 \\
0 & 0 & 0 & 10
\end{bmatrix}
\]
Solution

- Since the matrix is triangular then the eigenvalues are \( \lambda_1 = 1, \lambda_2 = 5, \lambda_3 = 8, \) and \( \lambda_4 = 10. \)
- Since there are four distinct eigenvalues there will be four linearly independent eigenvectors and thus the matrix is diagonalizable.
Geometric and Algebraic Multiplicity

**Remark:** an $n \times n$ matrix may be diagonalizable even if it does not have $n$ distinct **eigenvalues**. What is important is that it have $n$ linearly independent **eigenvectors**.
Geometric and Algebraic Multiplicity

**Remark:** an $n \times n$ matrix may be diagonalizable even if it does not have $n$ distinct **eigenvalues**. What is important is that it have $n$ linearly independent **eigenvectors**.

Suppose the characteristic polynomial of $A$ can be factored as

$$p(\lambda) = (\lambda - \lambda_1)^{n_1}(\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_k)^{n_k}$$
Remark: an $n \times n$ matrix may be diagonalizable even if it does not have $n$ distinct eigenvalues. What is important is that it have $n$ linearly independent eigenvectors.

Suppose the characteristic polynomial of $A$ can be factored as

$$p(\lambda) = (\lambda - \lambda_1)^{n_1}(\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_k)^{n_k}$$

then the eigenvalues of $A$ are $\lambda_1, \lambda_2, \ldots, \lambda_k$. 
**Remark:** an $n \times n$ matrix may be diagonalizable even if it does not have $n$ distinct eigenvalues. What is important is that it have $n$ linearly independent eigenvectors.

Suppose the characteristic polynomial of $A$ can be factored as

$$p(\lambda) = (\lambda - \lambda_1)^{n_1}(\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_k)^{n_k}$$

then the eigenvalues of $A$ are $\lambda_1, \lambda_2, \ldots, \lambda_k$.

- $n_i$ is called the **algebraic multiplicity** of $\lambda_i$.
- The dimension of the eigenspace corresponding to $\lambda_i$ is called the **geometric multiplicity** of $\lambda_i$. 
Theorem
If $A$ is a square matrix, then

1. The geometric multiplicity of every eigenvalue is less than or equal to its algebraic multiplicity.

2. $A$ is diagonalizable if and only if the geometric multiplicity of every eigenvalue is equal to its algebraic multiplicity.
Powers of a Matrix

If a square matrix $A$ is diagonalizable, then powers of $A$ are easy to compute.

$$D = P^{-1} AP$$
$$D^k = (P^{-1} AP)^k = \underbrace{(P^{-1} AP)(P^{-1} AP)\cdots(P^{-1} AP)}_{k \text{ factors}} = P^{-1} A(PP^{-1})A(P \cdots P^{-1})AP = P^{-1} A^k P$$
$$PD^k P^{-1} = A^k$$
Example

If \( A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix} \) then find \( A^{10} \).
Example

If $A = \begin{bmatrix} 3 & -1 & -2 \\
2 & 0 & -2 \\
2 & -1 & -1 \end{bmatrix}$ then find $A^{10}$.

We have already seen that if $P = \begin{bmatrix} 1 & 1 & 1 \\
1 & 2 & 0 \\
1 & 0 & 1 \end{bmatrix}$ with

$P^{-1} = \begin{bmatrix} -2 & 1 & 2 \\
1 & 0 & -1 \\
2 & -1 & -1 \end{bmatrix}$ then

$P^{-1}AP = D = \begin{bmatrix} 0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{bmatrix}$. 
Solution

\[ A^{10} = P D^{10} P^{-1} \]

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 0 \\
1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}^{10}
\begin{bmatrix}
-2 & 1 & 2 \\
1 & 0 & -1 \\
2 & -1 & -1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 0 \\
1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
-2 & 1 & 2 \\
1 & 0 & -1 \\
2 & -1 & -1
\end{bmatrix}
\]

\[
\begin{bmatrix}
3 & -1 & -2 \\
2 & 0 & -2 \\
2 & -1 & -1
\end{bmatrix}
\]
The converse of the diagonalizability theorem is false. There are $n \times n$ matrices with fewer than $n$ distinct eigenvalues that are diagonalizable.
Homework

- Read Section 5.2
- Exercises: 1, 3, 5, 10, 15, 20, 23, 24, 25.