Eigenvalues and Eigenvectors
MATH 322, Linear Algebra I

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Objectives

In this lesson we will define the terms **eigenvalue** and **eigenvector** and discuss some of their basic properties.
Definition
If $A$ is an $n \times n$ matrix and if there exists a nonzero vector $x \in \mathbb{R}^n$ and a scalar $\lambda$ such that $Ax = \lambda x$ then $\lambda$ is an eigenvalue of $A$ (or of $T_A$) and $x$ is said to be an eigenvector corresponding to $\lambda$.
Eigensystems

Definition
If $A$ is an $n \times n$ matrix and if there exists a nonzero vector $x \in \mathbb{R}^n$ and a scalar $\lambda$ such that $Ax = \lambda x$ then $\lambda$ is an 
\textit{eigenvalue} of $A$ (or of $T_A$) and $x$ is said to be an \textit{eigenvector} 
\textbf{corresponding to} $\lambda$.

Remarks:
\begin{itemize}
\item The zero vector $0$ is never an eigenvector since $A0 = \lambda 0$ 
\textbf{for all} $\lambda$.
\item Geometrically an eigenvalue/eigenvector pair means $Ax$ is 
a dilation/reflection of $x$ by scalar $\lambda$.
\end{itemize}
Example

Suppose \( A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \), then \( x = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \) is an eigenvector corresponding to \( \lambda = 3 \) since

\[
A x = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3x.
\]
Finding Eigenvalues

We must find nonzero solutions of

\[ A \mathbf{x} = \lambda \mathbf{x} \]

\[ A \mathbf{x} - \lambda \mathbf{x} = 0 \]

\[ \lambda \mathbf{x} - A \mathbf{x} = 0 \]

\[ \lambda \mathbf{x} - A \mathbf{x} = 0 \]

\[ (\lambda I - A)\mathbf{x} = 0 \]

\[ \implies \det(\lambda I - A) = 0 \]

Remarks:

- The expression \( \det(\lambda I - A) \) is called the characteristic polynomial of \( A \).

- The expression \( \det(\lambda I - A) = 0 \) is called the characteristic equation of \( A \).
Theorem

If $A$ is an $n \times n$ matrix, then $\lambda$ is an eigenvalue of $A$ if and only if it satisfies the characteristic equation of $A$

$$\det(\lambda I - A) = 0.$$
Example

Find all the eigenvalues of $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$.

$\det(\lambda I - A) = 0$

$\begin{vmatrix} \lambda - 3 & 0 \\ 8 & \lambda + 1 \end{vmatrix} = 0$

$$(\lambda - 3)(\lambda + 1) = 0$$

$\lambda = 3$ or $\lambda = -1$
Example

Find all the eigenvalues of \( A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \).

\[
\text{det}(\lambda I - A) = 0
\]

\[
\begin{vmatrix}
\lambda - 3 & 0 \\
-8 & \lambda + 1
\end{vmatrix} = 0
\]

\[
(\lambda - 3)(\lambda + 1) = 0
\]

\[
\lambda = 3 \quad \text{or} \quad \lambda = -1
\]
If $A$ is an $n \times n$ matrix then

$$\det(\lambda I - A) = \lambda^n + c_1 \lambda^{n-1} + \cdots + c_n = p(\lambda)$$

where $p$ is a polynomial.
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$$\det(\lambda I - A) = \lambda^n + c_1 \lambda^{n-1} + \cdots + c_n = p(\lambda)$$

where $p$ is a polynomial.

**Observation:** An $n \times n$ matrix will have $n$ eigenvalues. Some of these may be complex numbers.
Example

Find the eigenvalues of $A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$. 
Solution

\[ \det(\lambda I - A) = 0 \]

\[
\begin{vmatrix}
\lambda - 2 & 3 & -1 \\
-1 & \lambda + 2 & -1 \\
-1 & 3 & \lambda - 2
\end{vmatrix} = 0
\]
Solution

\[
det(\lambda I - A) = 0
\]
\[
\begin{vmatrix}
\lambda - 2 & 3 & -1 \\
-1 & \lambda + 2 & -1 \\
-1 & 3 & \lambda - 2
\end{vmatrix} = 0
\]
\[
\lambda^3 - 2\lambda^2 + \lambda = 0
\]
\[
\lambda(\lambda - 1)^2 = 0
\]
\[
\lambda = 0 \text{ or } \lambda = 1
\]
Theorem

If $A$ is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of $A$ are the diagonal entries of $A$. 
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If $A$ is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of $A$ are the diagonal entries of $A$.

Proof.

The determinant of a triangular matrix is the product of the diagonal entries.
Example

Find the eigenvalues of

\[
\begin{bmatrix}
3 & 1 & 7 & 11 \\
0 & 8 & 1 & \frac{1}{2} \\
0 & 0 & 0 & 2 \\
0 & 0 & 0 & 4
\end{bmatrix}.
\]
Example

Find the eigenvalues of

\[
\begin{bmatrix}
3 & 1 & 7 & 11 \\
0 & 8 & 1 & \frac{1}{2} \\
0 & 0 & 0 & 2 \\
0 & 0 & 0 & 4
\end{bmatrix}.
\]

\[
\begin{align*}
\lambda_1 &= 3 \\
\lambda_2 &= 8 \\
\lambda_3 &= 0 \\
\lambda_4 &= 4
\end{align*}
\]
Theorem
If $A$ is an $n \times n$ matrix and $\lambda \in \mathbb{R}$, then the following are equivalent.

1. $\lambda$ is an eigenvalue of $A$.
2. The system of equations $(\lambda I - A)x = 0$ has nontrivial solutions.
3. There exists a nonzero vector $x \in \mathbb{R}^n$ such that $Ax = \lambda x$.
4. $\lambda$ is a solution of the characteristic equation $\det(\lambda I - A) = 0$. 
Eigenspaces

Remark: The eigenvectors of $A \mathbf{x} = \lambda \mathbf{x}$ form a vector space called the eigenspace.
Remark: The eigenvectors of $A \mathbf{x} = \lambda \mathbf{x}$ form a vector space called the **eigenspace**.

Remarks: the eigenspace is

- the null space of $\lambda I - A$,
- the kernel of the matrix transformation $T_{\lambda I - A} : \mathbb{R}^n \to \mathbb{R}^n$,
- the set of vectors for which $A \mathbf{x} = \lambda \mathbf{x}$. 
Example

Previously we found the eigenvalues of $A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$ to be $\lambda_1 = 0$ and $\lambda_2 = \lambda_3 = 1$. Find the bases of the eigenspaces corresponding to each eigenvalue.
Solution (1 of 2)

- Take $\lambda_1 = 0$. In this case

$$\lambda_1 \mathbf{I} - \mathbf{A} = -\mathbf{A} = \begin{bmatrix} -2 & 3 & -1 \\ -1 & 2 & -1 \\ -1 & 3 & -2 \end{bmatrix}. $$

If we row reduce this matrix we have

$$\lambda_1 \mathbf{I} - \mathbf{A} \sim \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. $$

A basis for the eigenspace is \{ (1, 1, 1) \}.
Solution (1 of 2)

- Take $\lambda_1 = 0$. In this case

\[ \lambda_1 I - A = -A = \begin{bmatrix}
-2 & 3 & -1 \\
-1 & 2 & -1 \\
-1 & 3 & -2
\end{bmatrix}. \]

- If we row reduce this matrix we have

\[ \lambda_1 I - A \sim \begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & -1 \\
0 & 0 & 0
\end{bmatrix}. \]
Solution (1 of 2)

- Take $\lambda_1 = 0$. In this case
  \[
  \lambda_1 I - A = -A = \begin{bmatrix}
  -2 & 3 & -1 \\
  -1 & 2 & -1 \\
  -1 & 3 & -2 \\
  \end{bmatrix}.
  \]

- If we row reduce this matrix we have
  \[
  \lambda_1 I - A \sim \begin{bmatrix}
  1 & 0 & -1 \\
  0 & 1 & -1 \\
  0 & 0 & 0 \\
  \end{bmatrix}.
  \]

- A basis for the eigenspace is $\{(1, 1, 1)\}$. 
Solution (1 of 2)

- Take $\lambda_2 = \lambda_3 = 1$. In this case

$$\lambda_2 I - A = I - A \begin{bmatrix} -1 & 3 & -1 \\ -1 & 3 & -1 \\ -1 & 3 & -1 \end{bmatrix}.$$
Solution (1 of 2)

- Take $\lambda_2 = \lambda_3 = 1$. In this case

$$\lambda_2 I - A = I - A \begin{bmatrix}
-1 & 3 & -1 \\
-1 & 3 & -1 \\
-1 & 3 & -1 \\
\end{bmatrix}.$$  

- If we row reduce this matrix we have

$$\lambda_2 I - A \sim \begin{bmatrix}
1 & -3 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}.$$  

A basis for the eigenspace is

$$\{(1, 3, 0), (1, 0, -1)\}.$$
Take $\lambda_2 = \lambda_3 = 1$. In this case

$$\lambda_2 I - A = I - A \begin{bmatrix} -1 & 3 & -1 \\ -1 & 3 & -1 \\ -1 & 3 & -1 \end{bmatrix}.$$ 

If we row reduce this matrix we have

$$\lambda_2 I - A \sim \begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

A basis for the eigenspace is $\{(1, 3, 0), (1, 0, -1)\}$. 
Powers of a Matrix

Theorem
If $k$ is a positive integer and $\lambda$ is an eigenvalue of $A$ with corresponding eigenvector $\mathbf{x}$, then $\lambda^k$ is an eigenvalue of $A^k$ with $\mathbf{x}$ as its corresponding eigenvector.
Powers of a Matrix

Theorem
If \( k \) is a positive integer and \( \lambda \) is an eigenvalue of \( A \) with corresponding eigenvector \( \mathbf{x} \), then \( \lambda^k \) is an eigenvalue of \( A^k \) with \( \mathbf{x} \) as its corresponding eigenvector.

Proof.
Suppose \( A\mathbf{x} = \lambda \mathbf{x} \) then

\[
A^k \mathbf{x} = A^{k-1} (A \mathbf{x}) \\
= A^{k-1} \lambda \mathbf{x} \\
= \lambda A^{k-2} (A \mathbf{x}) \\
= \lambda^2 A^{k-2} \mathbf{x} \\
\vdots \\
A^k \mathbf{x} = \lambda^k \mathbf{x}.
\]
Eigenvalues and Invertibility

**Theorem**

*A square matrix $A$ is invertible if and only if $0$ is not an eigenvalue of $A$.*
Proof

- Note that if $A$ is an $n \times n$ matrix then $\lambda = 0$ is a solution to the characteristic equation

$$\lambda^n + c_1 \lambda^{n-1} + \cdots + c_{n-1} \lambda + c_n = 0$$

if and only if $c_n = 0$. Equivalently $A$ is invertible if and only if $c_n \neq 0$. 

Proof

- Note that if $A$ is an $n \times n$ matrix then $\lambda = 0$ is a solution to the characteristic equation

$$\lambda^n + c_1 \lambda^{n-1} + \cdots + c_{n-1} \lambda + c_n = 0$$

if and only if $c_n = 0$.

- Recall that

$$\det(\lambda I - A) = \lambda^n + c_1 \lambda^{n-1} + \cdots + c_{n-1} \lambda + c_n$$

so that

$$\det((0) I - A) = \det(-A) = (-1)^n \det(A) = c_n.$$
Proof

- Note that if $A$ is an $n \times n$ matrix then $\lambda = 0$ is a solution to the characteristic equation

$$\lambda^n + c_1 \lambda^{n-1} + \cdots + c_{n-1} \lambda + c_n = 0$$

if and only if $c_n = 0$.

- Recall that

$$\det(\lambda I - A) = \lambda^n + c_1 \lambda^{n-1} + \cdots + c_{n-1} \lambda + c_n$$

so that

$$\det((0) I - A) = \det(-A) = (-1)^n \det(A) = c_n.$$

- Thus $\det(A) = 0$ if and only if $c_n = 0$. Equivalently $A$ is invertible if and only if $c_n \neq 0$. 
Equivalent Statements

Theorem

If $A$ is an $n \times n$ matrix and if $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is multiplication by $A$, then the following are equivalent.

1. $A$ is invertible.
2. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
3. The reduced row echelon form of $A$ is $I_n$.
4. $A$ is expressible as the product of elementary matrices.
5. $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.
6. $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.
7. $\det(A) \neq 0$.
8. The column vectors of $A$ are linearly independent.
9. The row vectors of $A$ are linearly independent.
10. The column vectors of $A$ span $\mathbb{R}^n$. 
Theorem

If $A$ is an $n \times n$ matrix and if $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is multiplication by $A$, then the following are equivalent.

11. The row vectors of $A$ span $\mathbb{R}^n$.
12. The column vectors of $A$ form a basis for $\mathbb{R}^n$.
13. The row vectors of $A$ form a basis for $\mathbb{R}^n$.
14. $\text{rank}(A) = n$.
15. $\text{nullity}(A) = 0$.
16. The kernel of $T_A$ is $\{0\}$.
17. The range of $T_A$ is $\mathbb{R}^n$.
18. $T_A$ is one-to-one.
19. $\lambda = 0$ is not an eigenvalue of $A$. 
Linear Transformations

Definition
If $V$ and $W$ are vector spaces then the function $T : V \to W$ is a **linear transformation** if for all $u, v \in V$ and scalars $c$,

1. $T(u + v) = T(u) + T(v)$.
2. $T(c \cdot u) = c \cdot T(u)$.

If $V = W$, then $T$ is called a **linear operator** on $V$. 
Examples (1 of 3)

- If \( V \) and \( W \) are any two vector spaces the mapping 
  \( T : V \to W \) defined as \( T(v) = 0 \) for all \( v \in V \) is called the zero transformation.
Examples (1 of 3)

- If $V$ and $W$ are any two vector spaces the mapping $T : V \to W$ defined as $T(v) = 0$ for all $v \in V$ is called the zero transformation.

\[
T(u + v) = 0 = 0 + 0 = T(u) + T(v)
\]
\[
T(cu) = 0 = (c)0 = cT(u)
\]
Examples (1 of 3)

- If $V$ and $W$ are any two vector spaces the mapping $T : V \to W$ defined as $T(v) = 0$ for all $v \in V$ is called the zero transformation.

\[
T(u + v) = 0 = 0 + 0 = T(u) + T(v)
\]
\[
T(cu) = 0 = (c)0 = cT(u)
\]

- If $V$ is any vector space the mapping $T : V \to V$ defined as $T(v) = v$ for all $v \in V$ is called the
Examples (1 of 3)

- If $V$ and $W$ are any two vector spaces the mapping $T : V \to W$ defined as $T(v) = 0$ for all $v \in V$ is called the zero transformation.

  $T(u + v) = 0 = 0 + 0 = T(u) + T(v)$

  $T(cu) = 0 = (c)0 = cT(u)$

- If $V$ is any vector space the mapping $T : V \to V$ defined as $T(v) = v$ for all $v \in V$ is called the identity.

  $T(u + v) = u + v = T(u) + T(v)$

  $T(cu) = cu = cT(u)$
Examples (2 of 3)

- If $V$ is any vector space and $k$ is a fixed scalar, the mapping $T : V \rightarrow V$ defined as $T(v) = k \cdot v$ is a linear operator. If $k > 1$ this is called a **dilation** of $V$. If $0 < k < 1$ this is called a **contraction** of $V$. 

  $T(v + w) = k(v + w) = k \cdot v + k \cdot w = T(v) + T(w)$

  $T(c \cdot v) = k(c \cdot v) = c \cdot (k \cdot v) = c \cdot T(v)$
Examples (2 of 3)

If $V$ is any vector space and $k$ is a fixed scalar, the mapping $T : V \rightarrow V$ defined as $T(v) = k \cdot v$ is a linear operator. If $k > 1$ this is called a **dilation** of $V$. If $0 < k < 1$ this is called a **contraction** of $V$.

\[
T(u + v) = k(u + v) = k \cdot u + k \cdot v = T(u) + T(v)
\]

\[
T(c \cdot u) = k \cdot (c \cdot u) = c \cdot k \cdot u = c \cdot T(u)
\]
Examples (2 of 3)

- If $V$ is any vector space and $k$ is a fixed scalar, the mapping $T : V \rightarrow V$ defined as $T(v) = k\, v$ is a linear operator. If $k > 1$ this is called a **dilation** of $V$. If $0 < k < 1$ this is called a **contraction** of $V$.

\[
T(u + v) = k(u + v) = ku + kv = T(u) + T(v)
\]
\[
T(c\, u) = k\, cu = c\, ku = c\, T(u)
\]

- The mapping $T : P_n \rightarrow P_n$ given by $T(p) = T(p(x)) = p(x - 1)$ is a linear operator.
Examples (2 of 3)

- If \( V \) is any vector space and \( k \) is a fixed scalar, the mapping \( T : V \to V \) defined as \( T(v) = k \, v \) is a linear operator. If \( k > 1 \) this is called a **dilation** of \( V \). If \( 0 < k < 1 \) this is called a **contraction** of \( V \).

\[
T(u + v) = k(u + v) = ku + kv = T(u) + T(v)
\]
\[
T(c \, u) = k \, cu = c \, ku = c \, T(u)
\]

- The mapping \( T : P_n \to P_n \) given by \( T(p(x)) = p(x - 1) \) is a linear operator.

\[
T(p + q) = (p + q)(x - 1) = p(x - 1) + q(x - 1)
\]
\[
= T(p) + T(q)
\]
\[
T(c \, p) = (cp)(x - 1) = c \, p(x - 1) = c \, T(p)
\]
Examples (3 of 3)

- The mapping $D : C^1(-\infty, \infty) \to C(-\infty, \infty)$ defined by $D(f) = f'(x)$ is a linear transformation.
Examples (3 of 3)

- The mapping $D : C^1(-\infty, \infty) \rightarrow C(-\infty, \infty)$ defined by $D(f) = f'(x)$ is a linear transformation.

  
  \[
  D(f + g) = (f + g)'(x) = f'(x) + g'(x) = D(f) + D(g)
  
  D(cf) = (cf)'(x) = cf'(x) = c D(f)
  \]
Examples (3 of 3)

- The mapping $D : C^1(-\infty, \infty) \to C(-\infty, \infty)$ defined by $D(f) = f'(x)$ is a linear transformation.

  $$D(f + g) = (f + g)'(x) = f'(x) + g'(x) = D(f) + D(g)$$
  $$D(cf) = (cf)'(x) = cf'(x) = cD(f)$$

- The mapping $J : C(-\infty, \infty) \to C^1(-\infty, \infty)$ defined by $J(f) = \int_0^x f(t) \, dt$ is a linear transformation.

  $$J(f + g) = \int_0^x (f + g)(t) \, dt = \int_0^x f(t) \, dt + \int_0^t g(t) \, dt$$
  $$= J(f) + D(g)$$
  $$J(cf) = \int_0^x (cf)(x) = c \int_0^x f(t) \, dt = cJ(f)$$
Eigenvalues of a Linear Operator

Definition
If $T : V \to V$ is a linear operator on a vector space $V$, then a nonzero vector $x \in V$ is called an eigenvector of $T$ if $T(x)$ is a scalar multiple of $x$; that is,

$$T(x) = \lambda x$$

for some scalar $\lambda$. The scalar $\lambda$ is called an eigenvalue of $T$, and $x$ is said to be an eigenvector corresponding to $\lambda$. 
Definition

If $T : V \to V$ is a linear operator on a vector space $V$, then a nonzero vector $x \in V$ is called an eigenvector of $T$ if $T(x)$ is a scalar multiple of $x$; that is,

$$T(x) = \lambda x$$

for some scalar $\lambda$. The scalar $\lambda$ is called an eigenvalue of $T$, and $x$ is said to be an eigenvector corresponding to $\lambda$.

The subspace of all vectors in $V$ for which $T(x) = \lambda x$ is called the eigenspace of $T$ corresponding to $\lambda$. 
Example

If \( D : C^\infty(-\infty, \infty) \rightarrow C^\infty(-\infty, \infty) \) is the differentiation operator and if \( \lambda \) is a constant, then

\[
D(e^{\lambda x}) = \lambda e^{\lambda x}
\]

so that \( \lambda \) is an eigenvalue of \( D \) and \( e^{\lambda x} \) is the corresponding eigenvector.
Homework

- Read Section 5.1
- Exercises: 1–25 odd