Gaussian Elimination
MATH 322, *Linear Algebra I*

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Introduction

During this discussion we will develop a systematic (or algorithmic) procedure for solving a linear system of equations. We will use elementary row operations to convert an augmented matrix into an equivalent matrix representing a linear system whose solution can be found by inspection.
Definition
A matrix is in **row-echelon form** provided:

1. the first nonzero entry in a row (if any) is a 1. This is called a **leading 1**.
2. all rows containing only 0’s occur at the bottom of the matrix,
3. for any two adjacent nonzero rows, the leading 1 in the lower row occurs to the right of the leading 1 in the higher row.
Examples

Example
Which of the following matrices are in row-echelon form?

\[
\begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 2 & 3 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 2 \\
0 & 1 & 3 \\
0 & 1 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{bmatrix}
\]
Reduced Row-Echelon Form

Definition
A matrix in row-echelon form is said to be in **reduced row-echelon form** if each column containing a 1 has 0’s everywhere else.

Remark: Once an augmented matrix is in reduced row-echelon form, its solution is immediate.

\[
\begin{bmatrix}
1 & 0 & 0 & -2 \\
0 & 1 & 0 & 5 \\
0 & 0 & 1 & 4 \\
\end{bmatrix}
\]
Examples

Example
Which of the following matrices are in reduced row-echelon form?

\[
\begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1 \\
\end{bmatrix}
\quad
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 2 & 3 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\quad
\begin{bmatrix}
1 & 1 & 2 \\
0 & 1 & 3 \\
0 & 1 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{bmatrix}
\quad
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{bmatrix}
\]
Leading Variables and Free Variables

Definition
Unknowns corresponding to leading 1’s in the reduced row-echelon form of the augmented matrix are called **leading variables**. Unknowns without leading 1’s are called **free variables**.

Example
Given the augmented matrix

\[
\begin{bmatrix}
1 & 0 & 0 & 4 & -2 \\
0 & 1 & 0 & 5 & 6 \\
0 & 0 & 1 & 4 & 2
\end{bmatrix}
\]

write down the equivalent linear system, identify the leading and free variables, and find the solution to the linear system in terms of the free variables.
Gaussian Elimination will convert a given matrix to row-echelon form.

1. Find the leftmost column with a nonzero entry.
2. Swap the top row with another row to bring a nonzero entry to the top of the column found in Step 1.
3. If $a$ is the leftmost nonzero entry in the top row, multiply the row by $1/a$.
4. Add appropriate multiples of the current top row to the rows below so that zeros lie below the leading 1.
5. Repeat Steps 1–4 on the submatrix below the top row until the matrix is in row-echelon form.
Example

Perform Gaussian Elimination on the following matrix:

\[
\begin{bmatrix}
1 & -1 & 2 & -1 & -1 \\
2 & 1 & -2 & -2 & -2 \\
-1 & 2 & -2 & 1 & 1 \\
3 & 0 & 0 & -3 & -3 \\
\end{bmatrix}
\]
Gauss-Jordan Elimination

The procedure known as **Gauss-Jordan Elimination** will yield a matrix in reduced row-echelon form.

**Steps:**

1. Perform Gaussian Elimination until the matrix is in row-echelon form.
2. Beginning with the last nonzero row and working upward, add suitable multiples of that row to the rows above to introduce 0’s above the leading 1 of that row.
Example

Given the following matrix in row-echelon form, convert it to reduced row-echelon form and find the solution to the equivalent linear system.

\[
\begin{bmatrix}
1 & -1 & 2 & -1 & -1 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
Backwards Substitution

A matrix in row-echelon form can be used to solve an equivalent linear system using the procedure known as **backwards substitution**.

**Steps:**

1. Beginning with the bottom equation assign arbitrary values (called **parameters**) to any free variables.
2. Working from the bottom upwards solve successively for the leading variable in each row.
Example

Use backwards substitution to find the solution to the linear system equivalent to the following matrix in row-echelon form.

\[
\begin{bmatrix}
1 & -1 & 2 & -1 & -1 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
Homogeneous Linear Systems

Definition
A system of linear equations is **homogeneous** if the constant terms are all zero, \( i.e. \)

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
    \vdots & \\
    a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0
\end{align*}
\]

Remark: Every homogeneous system of equations is consistent since \( x_1 = x_2 = \cdots = x_n = 0 \) is a solution. There may be nontrivial solutions as well.
Nontrivial Solutions

Theorem
A homogeneous linear system with more unknowns than equations has infinitely many solutions.

Proof.
Suppose there are $m$ equations and $n$ unknowns with $m < n$. Suppose in the reduced row-echelon form the augmented matrix there are $r$ nonzero rows with $r \leq m < n$.

\[ \begin{align*}
\cdots x_{k_1} + \sum() &= 0 \\
\cdots x_{k_2} + \sum() &= 0 \\
\cdots \\
\cdots x_{k_r} + \sum() &= 0
\end{align*} \]
Homework

- Read Section 1.2
- Work exercises 1, 3, 5, 7, 12, 13, 17, 23–35 odd.