Further Results on Invertibility
MATH 322, Linear Algebra I

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Our discussion today will center on

▶ more results related to inverses of matrices,
▶ more results related to solving systems of linear equations.
Recall our graphical argument that a linear system of two equations in two unknowns has either no solution, a unique solution, or infinitely many solutions.
Basic Theorem

Recall our graphical argument that a linear system of two equations in two unknowns has either no solution, a unique solution, or infinitely many solutions.

Theorem

*Every system of linear equations has either no solutions, exactly one solution, or infinitely many solutions.*
One and only one of the following statements can be true of the number of solutions to the linear system $Ax = b$:

1. It has no solutions.
2. It has one solution.
3. It has more than one solution.

Our goal is to show that case 3 implies infinitely many solutions.
One and only one of the following statements can be true of the number of solutions to the linear system $Ax = b$:

1. It has no solutions.
2. It has one solution.
3. It has more than one solution.

Our goal is to show that case 3 implies infinitely many solutions.

If $Ax = b$ has two solutions, let them be $x_1$ and $x_2$. 

$A(x_1 - x_2) = b - b = 0$. 

Note that $x_0 = x_1 - x_2 
eq 0$. 

$A(x_0) = A(x_1 - x_2) = A(x_1) - A(x_2) = b - b = 0$. 

Thus, there are infinitely many solutions.
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Our goal is to show that case 3 implies infinitely many solutions.

- If $Ax = b$ has two solutions, let them be $x_1$ and $x_2$.
- Define $x_0 = x_1 - x_2 \neq 0$. 
Proof (1 of 2)

One and only one of the following statements can be true of the number of solutions to the linear system $Ax = b$:

1. It has no solutions.
2. It has one solution.
3. It has more than one solution.

Our goal is to show that case 3 implies infinitely many solutions.

► If $Ax = b$ has two solutions, let them be $x_1$ and $x_2$.
► Define $x_0 = x_1 - x_2 \neq 0$.
► Note that

$$Ax_0 = A(x_1 - x_2) = Ax_1 - Ax_2 = b - b = 0.$$
Let $k$ be any scalar and consider

\[ A(x_1 + kx_0) = Ax_1 + A(kx_0) = Ax_1 + k(Ax_0) = b + k(0) = b. \]

Thus $x_1 + kx_0$ is another solution to the linear system $Ax = b$. Since there are infinitely many choices for scalar $k$, there are infinitely many solutions.
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Thus $x_1 + kx_0$ is another solution to the linear system $Ax = b$. Since there are infinitely many choices for scalar $k$, there are infinitely many solutions.
Linear Systems

Theorem

If A is an invertible \( n \times n \) matrix, then for each \( n \times 1 \) matrix \( b \), the system of equations \( Ax = b \) has the unique solution \( x = A^{-1}b \).
Linear Systems

Theorem
If $A$ is an invertible $n \times n$ matrix, then for each $n \times 1$ matrix $b$, the system of equations $A x = b$ has the unique solution $x = A^{-1} b$.

Proof.

1. We can verify that $x = A^{-1} b$ is a solution, since

$$A(A^{-1} b) = (AA^{-1})b = b.$$ 

2. Let $x_0$ be any other solution to $A x = b$.

$$A x_0 = b$$

$$(A^{-1} A)x_0 = A^{-1} b$$

$$x_0 = A^{-1} b$$
Example

Using matrix inversion, find the solution to the system of equations below.

\[-x_1 + 4x_2 + x_3 = 0\]
\[x_1 + 9x_2 - 2x_3 = 1\]
\[6x_1 + 4x_2 - 8x_3 = 0\]
Example

Using matrix inversion, find the solution to the system of equations below.

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\[x_1 + 9x_2 - 2x_3 = 1\]
\[6x_1 + 4x_2 - 8x_3 = 0\]

If \( A = \begin{bmatrix} -1 & 4 & 1 \\ 1 & 9 & -2 \\ 6 & 4 & -8 \end{bmatrix} \) then \( A^{-1} = \begin{bmatrix} 32 & -18 & \frac{17}{2} \\ 2 & -1 & \frac{1}{2} \\ 25 & -14 & \frac{13}{2} \end{bmatrix} \)
Often we encounter related linear systems where only the right-hand side changes, \( i.e. \),

\[ Ax = b_1, \quad Ax = b_2, \quad \cdots, \quad Ax = b_k \]
Common Coefficient Matrices

Often we encounter related linear systems where only the right-hand side changes, \( i.e. \),

\[
Ax = b_1, \quad Ax = b_2, \quad \cdots, \quad Ax = b_k
\]

There are two means of solving these systems (of systems of equations):

1. Find \( A^{-1} \) and then calculate \( x_i = A^{-1}b_i \) for \( i = 1, 2, \ldots, k \).
2. Form the augmented matrix \[
\begin{bmatrix}
A & | & b_1 & | & b_2 & | & \cdots & | & b_k
\end{bmatrix}
\]
and use Gauss-Jordan elimination.
Often we encounter related linear systems where only the right-hand side changes, *i.e.*, 

\[ Ax = b_1, \quad Ax = b_2, \quad \cdots, \quad Ax = b_k \]

There are two means of solving these systems (of systems of equations):

1. Find \( A^{-1} \) and then calculate \( x_i = A^{-1}b_i \) for \( i = 1, 2, \ldots, k \).
2. Form the augmented matrix \([A|b_1|b_2|\cdots|b_k]\) and use Gauss-Jordan elimination.
Consider $Ax = b_i$ where

$$A = \begin{bmatrix} -1 & 4 & 1 \\ 1 & 9 & -2 \\ 6 & 4 & -8 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad b_2 = \begin{bmatrix} -3 \\ 4 \\ -5 \end{bmatrix}.$$

Use Gauss-Jordan elimination to solve both systems of equations.
Augmented form:

\[
\begin{bmatrix}
-1 & 4 & 1 & | & 0 & -3 \\
1 & 9 & -2 & | & 1 & 4 \\
6 & 4 & -8 & | & 1 & -5 \\
\end{bmatrix}
\]

Multiply the first row by \(-1\). Subtract the new first row from the second row. Multiply the new first row by \(-6\) and add to the third row.

\[
\begin{bmatrix}
1 & -4 & -1 & | & 0 & 3 \\
0 & 13 & -1 & | & 1 & 1 \\
0 & 28 & -2 & | & 1 & -23 \\
\end{bmatrix}
\]
Solution (2 of 3)

$$\begin{bmatrix} 1 & -4 & -1 & 0 & 3 \\ 0 & 13 & -1 & 1 & 1 \\ 0 & 28 & -2 & 1 & -23 \end{bmatrix}$$

Multiply the second row by $1/13$. Multiply the new second row by 4 and add to the first row. Multiply the new second row by $-28$ and add to the third row.

$$\begin{bmatrix} 1 & 0 & -17/13 & 4/13 & 43/13 \\ 0 & 1 & -1/13 & 1/13 & 1/13 \\ 0 & 0 & 2/13 & -15/13 & -327/13 \end{bmatrix}$$
Multiply the third row by $13/2$. Multiply the new third row by $17/13$ and add to the first row. Multiply the new third row by $1/13$ and add to the second row.

$$
\begin{bmatrix}
1 & 0 & 0 & -19/2 & -421/2 \\
0 & 1 & 0 & -1/2 & -25/2 \\
0 & 0 & 1 & -15/2 & -327/2
\end{bmatrix}
$$

First solution $x_1 = \begin{bmatrix} -19/2 \\ -1/2 \\ -15/2 \end{bmatrix}$ and second solution $x_2 = \begin{bmatrix} -421/2 \\ -25/2 \\ -327/2 \end{bmatrix}$
Properties of Invertible Matrices

Remark: we defined the inverse of matrix $A$ to be matrix $B$ such that $AB = I$ and $BA = I$. Actually only one condition is needed.
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**Theorem**

Let $A$ be an $n \times n$ matrix.

1. If $B$ is an $n \times n$ matrix such that $BA = I$, then $B = A^{-1}$.
2. If $B$ is an $n \times n$ matrix such that $AB = I$, then $B = A^{-1}$. 
Proof

- Suppose that $B$ is a square matrix and $BA = I$.
Proof

▶ Suppose that $B$ is a square matrix and $BA = I$.
▶ Let $x_0$ be any solution to $Ax = 0$.

\[
\begin{align*}
Ax_0 &= 0 \\
B(Ax_0) &= B(0) \\
(BA)x_0 &= 0 \\
x_0 &= 0
\end{align*}
\]

Thus $Ax = 0$ has only the trivial solution which implies $A$ is invertible.
Proof

- Suppose that $B$ is a square matrix and $BA = I$.
- Let $x_0$ be any solution to $Ax = 0$.

$$
\begin{align*}
Ax_0 &= 0 \\
B(Ax_0) &= B(0) \\
(BA)x_0 &= 0 \\
x_0 &= 0
\end{align*}
$$

Thus $Ax = 0$ has only the trivial solution which implies $A$ is invertible.

- Let the inverse of $A$ be $A^{-1}$.

$$
\begin{align*}
BAA^{-1} &= (BA)A^{-1} = IA^{-1} = A^{-1} \\
BAA^{-1} &= B(AA^{-1}) = BI = B
\end{align*}
$$
More Equivalence Statements

Theorem

If $A$ is an $n \times n$ matrix, then the following statements are equivalent:

1. $A$ is invertible.
2. $Ax = 0$ has only the trivial solution.
3. The reduced row echelon form of $A$ is $I_n$.
4. $A$ is expressible as the product of elementary matrices.
5. $Ax = b$ is consistent for every $n \times 1$ matrix $b$.
6. $Ax = b$ has exactly one solution for every $n \times 1$ matrix $b$. 
More Equivalence Statements

Theorem

If $A$ is an $n \times n$ matrix, then the following statements are equivalent:

1. $A$ is invertible.
2. $Ax = 0$ has only the trivial solution.
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5. $Ax = b$ is consistent for every $n \times 1$ matrix $b$.
6. $Ax = b$ has exactly one solution for every $n \times 1$ matrix $b$.

Previously we showed $1 \implies 2 \implies 3 \implies 4 \implies 1$.

An earlier theorem showed $1 \implies 6$. 
Proof ($6 \implies 5$)

- Assume $Ax = b$ has exactly one solution for every $n \times 1$ matrix $b$. 
Proof ($6 \implies 5$)

- Assume $Ax = b$ has exactly one solution for every $n \times 1$ matrix $b$.
- By definition, then $Ax = b$ is consistent for every $n \times 1$ matrix $b$. 
Assume $Ax = b$ is consistent for every $n \times 1$ matrix $b$. The following systems are then consistent:

\[
Ax = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \quad Ax = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.
\]
Proof $ (5 \implies 1)$ (1 of 2)

- Assume $Ax = b$ is consistent for every $n \times 1$ matrix $b$.
- The following systems are then consistent.

\[
Ax = \begin{bmatrix}
1 \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix}, \quad Ax = \begin{bmatrix}
0 \\
1 \\
0 \\
\vdots \\
0
\end{bmatrix}, \ldots, \quad Ax = \begin{bmatrix}
0 \\
0 \\
0 \\
\vdots \\
1
\end{bmatrix}.
\]

- Let $x_1, x_2, \ldots, x_n$ be the respective solutions.
Proof $(5 \implies 1)$ (2 of 2)

Define the matrix $C$ as

$$C = [x_1 \mid x_2 \mid \cdots \mid x_n].$$
Proof (5 \implies 1) (2 of 2)

Define the matrix $C$ as

$$C = [x_1 | x_2 | \cdots | x_n].$$

Calculate $AC$:

$$AC = \begin{bmatrix} Ax_1 & Ax_2 & \cdots & Ax_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I$$

Thus $C = A^{-1}$ and $A$ is invertible.
Proof (5 $\implies$ 1) (2 of 2)

- Define the matrix $C$ as
  \[ C = [\mathbf{x}_1 \mid \mathbf{x}_2 \mid \cdots \mid \mathbf{x}_n]. \]

- Calculate $AC$:
  \[
  AC = [A\mathbf{x}_1 \mid A\mathbf{x}_2 \mid \cdots \mid A\mathbf{x}_n] = \begin{bmatrix}
  1 & 0 & \cdots & 0 \\
  0 & 1 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & 1
  \end{bmatrix} = I
  \]

- Thus $C = A^{-1}$ and $A$ is invertible.
Recall: we have seen that if $n \times n$ matrices $A$ and $B$ are invertible then $AB$ is invertible.
Recall: we have seen that if $n \times n$ matrices $A$ and $B$ are invertible then $AB$ is invertible.

The converse is also true.

**Theorem**

Let $A$ and $B$ be $n \times n$ matrices. If $AB$ is invertible, then $A$ and $B$ are invertible.
Proof

Let $x_0$ be any solution of the linear system $Bx = 0$. Then

$$(AB)x_0 = A(Bx_0) = A(0) = 0.$$
Proof

- Let $x_0$ be any solution of the linear system $Bx = 0$. Then

  \[(AB)x_0 = A(Bx_0) = A(0) = 0.\]

- Since $AB$ is invertible then $x_0 = 0$.
Proof

- Let \( x_0 \) be any solution of the linear system \( Bx = 0 \). Then
  \[
  (AB)x_0 = A(Bx_0) = A(0) = 0.
  \]

- Since \( AB \) is invertible then \( x_0 = 0 \).

- Since \( x_0 = 0 \) is the only solution to \( Bx = 0 \), then \( B \) is invertible. Consequently \( B^{-1} \) is invertible.
Proof

- Let $x_0$ be any solution of the linear system $Bx = 0$. Then
  \[(AB)x_0 = A(Bx_0) = A(0) = 0.\]

- Since $AB$ is invertible then $x_0 = 0$.

- Since $x_0 = 0$ is the only solution to $Bx = 0$, then $B$ is invertible. Consequently $B^{-1}$ is invertible.

- $A$ is invertible since
  \[A = AI = A(BB^{-1}) = (AB)B^{-1}\]
  it is the product of invertible matrices.
Fundamental Problem

**Problem:** let $A$ be a fixed $m \times n$ matrix. Find all the $m \times 1$ matrices $b$ such that the system of equations $Ax = b$ is consistent.
Example

Determine, by row-reducing the augmented matrix, the conditions which $b_1$, $b_2$, and $b_3$ must satisfy for the following system to be consistent.

\begin{align*}
x_1 + x_2 + 2x_3 &= b_1 \\
x_1 + x_3 &= b_2 \\
2x_1 + x_2 + 3x_3 &= b_3
\end{align*}
Solution

Augmented matrix:

\[
\begin{bmatrix}
1 & 1 & 2 & b_1 \\
1 & 0 & 1 & b_2 \\
2 & 1 & 3 & b_3 \\
\end{bmatrix}
\]

System is consistent only if \( b_3 = b_1 + b_2 \).
Solution

Augmented matrix:

\[
\begin{bmatrix}
1 & 1 & 2 & b_1 \\
1 & 0 & 1 & b_2 \\
2 & 1 & 3 & b_3 \\
\end{bmatrix}
\]

Row reduced form:

\[
\begin{bmatrix}
1 & 1 & 2 & b_1 \\
0 & 1 & 1 & b_1 - b_2 \\
0 & 0 & 0 & b_3 - b_2 - b_1 \\
\end{bmatrix}
\]

System is consistent only if \( b_3 = b_1 + b_2 \).
Solution

Augmented matrix:

\[
\begin{bmatrix}
1 & 1 & 2 & b_1 \\
1 & 0 & 1 & b_2 \\
2 & 1 & 3 & b_3 \\
\end{bmatrix}
\]

Row reduced form:

\[
\begin{bmatrix}
1 & 1 & 2 & b_1 \\
0 & 1 & 1 & b_1 - b_2 \\
0 & 0 & 0 & b_3 - b_2 - b_1 \\
\end{bmatrix}
\]

System is consistent only if \( b_3 = b_1 + b_2 \).
Homework

- Read Section 1.6
- Work exercises 1–5, 9, 16, 21–24.