Matrix Transformations
MATH 322, Linear Algebra I

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Introduction

Today we will discuss the behavior and properties of functions whose domain is a subset of the vectors in \( \mathbb{R}^n \) and whose range is a subset of the vectors in \( \mathbb{R}^m \).

Remarks:

- Linear transformations are of fundamental importance in science, engineering, and mathematics.
- Most of our examples will be anchored in two- and three-dimensional spaces.
Functions

Definition
A function $f$ is a rule of correspondence that associates to each element $a \in A$ (the domain set) a unique element $b \in B$ (the codomain).

\[
\text{range} = \{f(a) | a \in A\}
\]
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\]

Remarks:
- Two functions $f_1$ and $f_2$ are **equal** if they have the same domain and if $f_1(a) = f_2(a), \forall a \in A$.
- When the domain is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$ we call $f$ a **map** or **transformation** from $\mathbb{R}^n$ to $\mathbb{R}^m$. $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called an **operator**.
Maps

Generically a map \( f : \mathbb{R}^n \to \mathbb{R}^m \) can be expressed as

\[
\begin{align*}
\mathbf{w}_1 & = f_1(x_1, x_2, \ldots, x_n) \\
\mathbf{w}_2 & = f_2(x_1, x_2, \ldots, x_n) \\
\vdots & \\
\mathbf{w}_m & = f_m(x_1, x_2, \ldots, x_n)
\end{align*}
\]
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    w_2 &= f_2(x_1, x_2, \ldots, x_n) \\
    &\vdots \\
    w_m &= f_m(x_1, x_2, \ldots, x_n)
\end{align*}
\]

If $f_1, f_2, \ldots, f_m$ are linear functions of $x_1, x_2, \ldots, x_n$ then the transformation is called \textbf{linear}. 
Matrix Multiplication

If we name the transformation \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) then \( A \) is said to be the **standard matrix** for the linear transformation.

\( T \) is multiplication by \( A \).
Examples (1 of 2)

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is such that $T(x) = 0$ for all $x \in \mathbb{R}^n$ then

$$T(x) = T_0(x) = 0x = 0$$

is called the **zero transformation**.
If $T : \mathbb{R}^n \to \mathbb{R}^n$ is such that $T(x) = x$ for all $x \in \mathbb{R}^n$ then

$$T(x) = T_I(x) = I_n x = x$$

is called the identity transformation.
Two-Dimensional Reflection Operators (1 of 3)

Suppose $T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, then

$T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} -x \\ y \end{bmatrix}$ (reflection about $y$-axis).
Two-Dimensional Reflection Operators (2 of 3)

Suppose \( T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \), then

\[ T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x \\ -y \end{bmatrix} \] (reflection about \( x \)-axis).
Suppose $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, then

$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$ (reflection about $y = x$).
Reflection about xy-plane,

\[ T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \]
Three-Dimensional Reflection Operators (2 of 3)

Reflection about $xz$-plane,

\[
T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.
\]
Three-Dimensional Reflection Operators (3 of 3)

Reflection about $yz$-plane,

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$
Definition
A projection operator (sometimes called an orthogonal projection operator) maps a vector to its orthogonal projection on a line or plane through the origin.
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A **projection operator** (sometimes called an **orthogonal projection operator**) maps a vector to its orthogonal projection on a line or plane through the origin.

- Projection onto the $x$-axis, $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

- Projection onto the $y$-axis, $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.
Pictures

Project onto x-axis

\[ T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \]

Project onto x-axis

\[ T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \]
Three-dimensional Projections

- **Projection onto $xy$-plane,**
  \[
  T \begin{pmatrix}
  x \\
  y \\
  z
  \end{pmatrix} = \begin{pmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 0
  \end{pmatrix} \begin{pmatrix}
  x \\
  y \\
  z
  \end{pmatrix} = \begin{pmatrix}
  x \\
  y \\
  0
  \end{pmatrix}.
  \]

- **Projection onto $xz$-plane,**
  \[
  T \begin{pmatrix}
  x \\
  y \\
  z
  \end{pmatrix} = \begin{pmatrix}
  1 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 1
  \end{pmatrix} \begin{pmatrix}
  x \\
  y \\
  z
  \end{pmatrix} = \begin{pmatrix}
  x \\
  0 \\
  z
  \end{pmatrix}.
  \]

- **Projection onto $yz$-plane,**
  \[
  T \begin{pmatrix}
  x \\
  y \\
  z
  \end{pmatrix} = \begin{pmatrix}
  0 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
  \end{pmatrix} \begin{pmatrix}
  x \\
  y \\
  z
  \end{pmatrix} = \begin{pmatrix}
  0 \\
  y \\
  z
  \end{pmatrix}.
  \]
Pictures

Projection on $xy$-plane

Projection on $xz$-plane

Projection on $yz$-plane
Rotation Operators

**Definition**
A operator that rotates a vector through a fixed angle $\theta$ is called a rotation operator.
Rotation Operators

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Theorem
The standard matrix for the rotation operator on $\mathbb{R}^2$ which rotates a vector counterclockwise through an angle $\theta$ is

$$
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
$$
\[ w_1 = x \cos \theta - y \sin \theta \]
\[ w_2 = x \sin \theta + y \cos \theta \]
Example

Suppose the vector \( \mathbf{x} = \left( \frac{\sqrt{3}}{2}, \frac{1}{2} \right) \) is rotated \( \theta = 45^\circ \) counterclockwise around the origin.

\[
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
\frac{\sqrt{3}}{2} \\
\frac{1}{2}
\end{bmatrix}
= \frac{\sqrt{2}}{2}
\begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
\frac{\sqrt{3}}{2} \\
\frac{1}{2}
\end{bmatrix}
= \begin{bmatrix}
\frac{\sqrt{3} - 1}{2\sqrt{2}} \\
\frac{\sqrt{2} + \sqrt{6}}{4}
\end{bmatrix}
\]
Rotations in $\mathbb{R}^3$

To rotate a vector in $\mathbb{R}^3$ we will make note of

- a line through the origin called the **axis of rotation**,  
- a unit vector $\mathbf{u}$ along the axis of rotation,  
- the **right-hand rule** to determine if the rotation is **positive** or **negative**.

Curl the fingers of your right hand around the axis of rotation in the direction of the rotation. If your thumb points in the same direction as $\mathbf{u}$, the rotation is **positive**, otherwise it is **negative**.
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- Curl the fingers of your right hand around the axis of rotation in the direction of the rotation. If your thumb points in the same direction as $u$, the rotation is positive, otherwise it is negative.
Right-hand Rule
Rotations Around Positive $x$-axis

Counterclockwise rotation about the positive $x$-axis through angle $\theta$:

**Illustration**

**Standard Matrix**

$$\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{bmatrix}$$
Rotations Around Positive $y$-axis

Counterclockwise rotation about the positive $y$-axis through angle $\theta$:

**Illustration**

**Standard Matrix**

\[
\begin{bmatrix}
\cos \theta & 0 & \sin \theta \\
0 & 1 & 0 \\
-\sin \theta & 0 & \cos \theta
\end{bmatrix}
\]
Rotations Around Positive $z$-axis

Counterclockwise rotation about the positive $z$-axis through angle $\theta$:

**Standard Matrix**

\[
\begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
Dilation and Contraction Operators

Dilation and contraction operators stretch or compress vectors without rotation, reflection, or translation.

\[ T(x) = kx \]

where the action is
- contraction, if \( 0 \leq k < 1 \)
- dilation, if \( 1 < k \)
Compositions of Linear Transformations

Suppose $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^k$ and $T_B : \mathbb{R}^k \rightarrow \mathbb{R}^m$ then

$$(T_B \circ T_A) : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

where

$$(T_B \circ T_A)(x) = T_B(T_A(x)) = T_B(Ax) = B(Ax) = (BA)x.$$
Compositions of Linear Transformations

Suppose $T_A : \mathbb{R}^n \to \mathbb{R}^k$ and $T_B : \mathbb{R}^k \to \mathbb{R}^m$ then $(T_B \circ T_A) : \mathbb{R}^n \to \mathbb{R}^m$ where

$$(T_B \circ T_A)(x) = T_B(T_A(x)) = T_B(Ax) = B(Ax) = (BA)x.$$

Remarks:

- Composition of linear transformations is equivalent to multiplying the standard matrices representing the transformations.

$T_B \circ T_A = T_{BA}$

- Since matrix multiplication is not commutative, then composition of linear transformations is not commutative.
Rotate the vector \( \mathbf{x} = \left( \frac{\sqrt{3}}{2}, \frac{1}{2} \right) \) by \( \theta = \frac{3\pi}{4} \) counterclockwise around the origin and project it onto the \( x \)-axis.
Example (2 of 2)

Find the standard matrix for the transformation which reflects a vector about the line through the origin making an angle of $\pi/6$ with the positive $x$-axis.
Example (2 of 2)

Find the standard matrix for the transformation which reflects a vector about the line through the origin making an angle of $\pi/6$ with the positive $x$-axis.

We can decompose this transformation into three operations.

1. Rotate counterclockwise by $\pi/3$.
2. Reflect about the $y$-axis.
3. Rotate clockwise by $\pi/3$. 
Solution

\[
T = \begin{bmatrix}
\cos \frac{-\pi}{3} & -\sin \frac{-\pi}{3} \\
\sin \frac{-\pi}{3} & \cos \frac{-\pi}{3}
\end{bmatrix}
\begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\cos \frac{\pi}{3} & -\sin \frac{\pi}{3} \\
\sin \frac{\pi}{3} & \cos \frac{\pi}{3}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\frac{1}{2} & \frac{\sqrt{3}}{2} \\
-\frac{\sqrt{3}}{2} & \frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\frac{1}{2} & -\frac{\sqrt{3}}{2} \\
\frac{\sqrt{3}}{2} & \frac{1}{2}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\frac{1}{2} & \frac{\sqrt{3}}{2} \\
\frac{\sqrt{3}}{2} & \frac{1}{2}
\end{bmatrix}
\]
Homework

- Read Section 4.9
- Exercises: 1–19 odd