Matrices and Matrix Operations
MATH 322, Linear Algebra I

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Definition
A matrix is any rectangular array of numbers.

- An $n \times m$ matrix contains $n$ rows and $m$ columns.
- A $1 \times m$ matrix is called a row vector.
- A $n \times 1$ matrix is called a column vector.
- A $1 \times 1$ matrix is called a scalar.

Notation: $A = (a_{ij})_{n \times m}$
$a_{ij} = (A)_{ij}$ is the entry in the $i^{th}$ row and $j^{th}$ column of matrix $A$. 
Square Matrices

Definition

A square matrix of order $n$ is a matrix with $n$ rows and $n$ columns.

Definition

The main diagonal of a square matrix of order $n$ is the entries \( \{a_{11}, a_{22}, \ldots, a_{nn}\} \).

\[
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\]
Matrix Operations

Definition
Two matrices \( A \) and \( B \) are **equal**, denoted \( A = B \) if they have the same size and \( a_{ij} = b_{ij} \) for all \( i \) and \( j \).

Definition
If matrices \( A \) and \( B \) have the same size then the **sum of \( A \) and \( B \)** denoted \( A + B \) is a matrix obtained by adding corresponding entries in matrices \( A \) and \( B \).

\[
(A + B)_{ij} = (A)_{ij} + (B)_{ij} = a_{ij} + b_{ij}
\]
Matrix Operations (continued)

Definition
If matrices $A$ and $B$ have the same size then the **difference of $A$ and $B$** denoted $A - B$ is a matrix obtained by subtracting corresponding entries in matrix $B$ from those in matrix $A$.

$$(A - B)_{ij} = (A)_{ij} - (B)_{ij} = a_{ij} - b_{ij}$$

Example
Given

$$A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -4 & 3 & 5 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & 2 & -4 & 5 \end{bmatrix}$$

find $A + B$ and $A - B$. 
Scalar Product

Definition
If $A$ is a matrix and $c$ is a scalar, the **scalar product of $c$ and $A$** denoted $cA$ is a matrix obtained by multiplying each entry of $A$ by $c$.

$$(cA)_{ij} = c(A)_{ij} = ca_{ij}$$

Definition
If $A_1, A_2, \ldots, A_n$ are all matrices of the same size and if $c_1, c_2, \ldots, c_n$ are scalars then an expression of the form

$$c_1 A_1 + c_2 A_2 + \cdots + c_n A_n$$

is called a **linear combination** of $A_1, A_2, \ldots, A_n$ with **coefficients** $c_1, c_2, \ldots, c_n$.

$$(c_1 A_1 + c_2 A_2 + \cdots + c_n A_n)_{ij} = c_1 (A_1)_{ij} + c_2 (A_2)_{ij} + \cdots + c_n (A_n)_{ij}$$
Examples

Example

Given

\[ A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -4 & 3 & 5 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & 2 & -4 & 5 \end{bmatrix} \]

find \( 3A \) and \( 2A - 3B \).
Matrix Multiplication

Definition
If $A$ is an $m \times p$ matrix and $B$ is a $p \times n$ matrix, the product of $A$ and $B$ is an $m \times n$ matrix where

$$(AB)_{ij} = \sum_{k=1}^{p} (A)_{ik}(B)_{kj} = \sum_{k=1}^{p} a_{ik}b_{kj}$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Example
Given

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 & 5 \\ 2 & 4 & 1 \end{bmatrix}$$

find $AB$. 
Examples

Example
Suppose that matrix $A$ has size $3 \times 4$, matrix $B$ has size $7 \times 3$, and matrix $C$ has size $4 \times 7$. Which pairs of matrix products are defined and what are the sizes of the products?
Multiplication by Rows and Columns

\[ AB = \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1r} \\
    a_{21} & a_{22} & \cdots & a_{2r} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{i1} & a_{i2} & \cdots & a_{ir} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mr}
\end{bmatrix} \begin{bmatrix}
    b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\
    b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\
    \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
    b_{r1} & b_{r2} & \cdots & b_{rj} & \cdots & b_{rn}
\end{bmatrix} \]

Note:

\[(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ir}b_{rj}\]

Observations:

- \(j^{th}\) column of \(AB = A[j^{th}\text{column of } B]\)
- \(i^{th}\) row of \(AB = [i^{th}\text{row of } A]B\)
Example

Given

\[ A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \] and \[ B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} \]

find the second row of \( AB \) and the third column of \( AB \).
A matrix can be **partitioned** by dividing it into submatrices. Of particular importance are the partitions into rows and columns. Suppose

\[
A = \begin{bmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} \\
  a_{21} & a_{22} & a_{23} & a_{24} \\
  a_{n1} & a_{n2} & a_{33} & a_{34}
\end{bmatrix}
\]

then

\[
A = \begin{bmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} \\
  a_{21} & a_{22} & a_{23} & a_{24} \\
  a_{n1} & a_{n2} & a_{33} & a_{34}
\end{bmatrix} = \begin{bmatrix}
  r_1 \\
  r_2 \\
  r_3
\end{bmatrix}
\]

and

\[
A = \begin{bmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} \\
  a_{21} & a_{22} & a_{23} & a_{24} \\
  a_{n1} & a_{n2} & a_{33} & a_{34}
\end{bmatrix} = \begin{bmatrix}
  c_1 & c_2 & c_3 & c_4
\end{bmatrix}
\]
Matrix Products by Partitioning

Let $a_1, a_2, \ldots, a_m$ be the row matrices of matrix $A$ and let $b_1, b_2, \ldots, b_n$ be the column matrices of matrix $B$ then we may compute the product $AB$ in the following ways:

- Column by column

$$AB = A \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix} = \begin{bmatrix} Ab_1 & Ab_2 & \cdots & Ab_n \end{bmatrix}$$

- Row by row

$$AB = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} B = \begin{bmatrix} a_1 B \\ a_2 B \\ \vdots \\ a_m B \end{bmatrix}$$
Linear Combinations

Suppose $A$ is an $m \times n$ matrix and $x$ is a $n \times 1$ column vector, then

$$Ax = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$
Remarks

- The product $Ax$ where $A$ is an $m \times n$ matrix and $x$ is an $n \times 1$ column vector is a linear combination of the columns of $A$ with the entries of $x$ serving as coefficients.

- The product $yA$ where $A$ is an $m \times n$ matrix and $y$ is an $1 \times m$ row vector is a linear combination of the rows of $A$ with the entries of $y$ serving as coefficients.
Consider the linear system of $m$ equations with $n$ unknowns.

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
    &\vdots \\
    a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{align*}
\]

Think of each side of the equation as a column matrix.

\[
\begin{bmatrix}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
    &\vdots \\\n    a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n
\end{bmatrix}
\]
Linear Systems (continued)

Think of the column matrix as a product.

\[
\begin{bmatrix}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \\
\end{bmatrix}
= 
\begin{bmatrix}
b_1 \\
b_2 \\
\vdots \\
b_m \\
\end{bmatrix}
= 
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \cdots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn} \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n \\
\end{bmatrix}
= 
\begin{bmatrix}
b_1 \\
b_2 \\
\vdots \\
b_m \\
\end{bmatrix}
\]

\[Ax = b\]
Transposes

Definition
If $A$ is an $m \times n$ matrix, the transpose of $A$ denoted $A^T$ is an $n \times m$ matrix such that

$$(A^T)_{ij} = (A)_{ji}$$

for all $i$ and $j$.

Example
Given

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \\ 2 & -1 \end{bmatrix}$$

find $A^T$. 
Definition
If $A$ is a square matrix then the trace of $A$ denoted $\text{tr}(A)$ is the sum of the entries on the diagonal of $A$.

Example
Given

\[
A = \begin{bmatrix}
5 & 5 & 16 \\
6 & 12 & 3 \\
4 & -2 & 9
\end{bmatrix}
\]

find $\text{tr}(A)$.
Homework

- Read Section 1.3
- Work exercises 1, 3, 7, 9, 12, 13, 18, 21, 23, 24, 25–31 odd.