Rules for Matrix Arithmetic
MATH 322, Linear Algebra I

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Introduction

- Laws of matrix arithmetic
- Many laws from ordinary (scalar) arithmetic carry over
- Several common ones do not.
- We will assume the sizes of matrices are such that the operations described can be carried out.
Properties of Matrix Arithmetic

Theorem

Let $A$, $B$, and $C$ be matrices, then

- $A + B = B + A$ (Commutative Law of Addition)
- $A + (B + C) = (A + B) + C$ (Associative Law of Addition)
- $A(BC) = (AB)C$ (Associative Law of Multiplication)
- $A(B + C) = AB + AC$ (Left Distributive Law)
- $(B + C)A = BA + CA$ (Right Distributive Law)
Properties of Matrix Arithmetic

Theorem

Let $A$, $B$, and $C$ be matrices and $a$ and $b$ be scalars, then

- $A(B - C) = AB - AC$
- $(B - C)A = BA - CA$
- $a(B + C) = aB + aC$
- $a(B - C) = aB - aC$
- $(a + b)C = aC + bC$
- $(a - b)C = aC - bC$
- $a(bC) = (ab)C$
- $a(BC) = (aB)C = B(aC)$
Question: Which common property is missing from the theorem?
Missing Properties

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**Answer:** Commutativity of matrix multiplication.
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Answer: Commutativity of matrix multiplication.

Example
Let
\[
A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix},
\]
then
\[
AB = \begin{bmatrix} 4 & 2 \\ 4 & 1 \end{bmatrix} \quad \text{but} \quad BA = \begin{bmatrix} 5 & 1 \\ 4 & 0 \end{bmatrix}.
\]
Zero Matrices

Definition
A matrix all of whose entries are 0 is called a zero matrix.

Notation: the zero matrix will be denoted 0.

Theorem
Let $A$ be an arbitrary matrix and let $0$ be a zero matrix, then

- $A + 0 = 0 + A = A$ (Additive Identity)
- $A - A = 0$ (Additive Inverse)
- $0 - A = -A$
- $A0 = 0A = 0$
For **scalar arithmetic** we are accustomed to the **cancellation law**:

\[
\text{If } ab = ac \text{ and } a \neq 0 \text{ then } b = c.
\]

This does not hold true for matrices.

**Example**

\[
\begin{bmatrix}
0 & 1 \\
0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
0 & 0 \\
\end{bmatrix}
=
\begin{bmatrix}
0 & 1 \\
0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
0 & 2 \\
0 & 0 \\
\end{bmatrix}
=
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
\end{bmatrix}
\]
For **scalar arithmetic** we are accustomed to the **zero factor property**:  

\[ \text{If } ab = 0 \text{ then } a = 0 \text{ or } b = 0. \]

This does not hold true for matrices.

**Example**

\[
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
= 
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\]
Identity Matrices

Definition
A square matrix with 1’s on the diagonal and 0’s elsewhere is called an **identity matrix**.

Notation: the $n \times n$ identity matrix will be denoted $I_n$ or merely $I$ (when the size is understood from the context).

Theorem
If $A$ is an $m \times n$ matrix then $AI_n = A$ and $I_mA = A$.

Remark: Identity matrices play the role of the multiplicative identity element for matrix multiplication.
Theorem

If $A$ is an $n \times n$ matrix and $R$ is its reduced row-echelon form, then $R$ either contains a row of 0’s or is $I_n$. 
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If $A$ is an $n \times n$ matrix and $R$ is its reduced row-echelon form, then $R$ either contains a row of 0’s or is $I_n$.

Proof.
Let the reduced row-echelon form of $A$ be

\[
R = \begin{bmatrix}
    r_{11} & r_{12} & \cdots & r_{1n} \\
    r_{21} & r_{22} & \cdots & r_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    r_{n1} & r_{n2} & \cdots & r_{nn}
\end{bmatrix}
\]
1. Either the last row contains only 0’s or it does not.
Proof (continued)

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2. If the matrix has no row of 0’s, each row has a leading entry of 1.
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2. If the matrix has no row of 0’s, each row has a leading entry of 1.
3. The leading 1’s occur along the main diagonal and the entries in the same column as a 1 must be 0.
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4. Thus if matrix $R$ has no row of 0’s then $R = I_n$. 
Matrix Inverses

Definition
If $A$ and $B$ are $n \times n$ matrices such that $AB = BA = I_n$ then $A$ is invertible and $B$ is the inverse of $A$. If $A$ has no inverse then $A$ is said to be singular.

Notation: the inverse of an invertible matrix $A$ will be denoted $A^{-1}$.

Example

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$, then

$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$
Example

Let \( A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \) and \( B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \), then

\[
BA = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b_{11} + b_{12} & 0 \\ b_{21} + b_{22} & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2
\]
Uniqueness of an Inverse

Theorem

*If B and C are both inverses of A then \( B = C \).*
Uniqueness of an Inverse

Theorem
If $B$ and $C$ are both inverses of $A$ then $B = C$.

Proof.

\[
BA = I \\
(BA)C = IC = C \\
B(AC) = BI = B \quad \text{(associativity)} \\
B = C
\]
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Remark: the inverse of a matrix (if it exists) is unique.
Theorem
If $A$ is the $2 \times 2$ matrix with

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then $A$ is invertible if $ad - bc \neq 0$, in which case

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$
Properties of Inverses

Theorem
If $A$ and $B$ are $n \times n$ invertible matrices then $AB$ is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$
Properties of Inverses

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If $A$ and $B$ are $n \times n$ invertible matrices then $AB$ is invertible and

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Proof.

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I$$
Properties of Inverses

Theorem

If $A$ and $B$ are $n \times n$ invertible matrices then $AB$ is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$  

Proof.

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$$

Remark: the product of any number of invertible matrices is invertible and the inverse of the product is the product of the inverses in reverse order.
Properties of Inverses

Theorem

If $A$ and $B$ are $n \times n$ invertible matrices then $AB$ is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$ 

Proof.

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Remark: the product of any number of invertible matrices is invertible and the inverse of the product is the product of the inverses in reverse order.
Powers of a Matrix

Definition
If $A$ is an $n \times n$ matrix then we define the nonnegative powers of $A$ as follows.

$$A^0 = I_n \quad \text{and} \quad A^n = AA \cdots A \quad \text{for } n > 0.$$ $n$ factors

If $A$ is an invertible matrix then the negatives powers of $A$ are defined as

$$A^{-n} = (A^{-1})^n = A^{-1}A^{-1} \cdots A^{-1}.$$ $n$ factors
Laws of Exponents

Theorem
If $A$ is a square matrix and $r$ and $s$ are nonnegative integers then $A^r A^s = A^{r+s}$.

Theorem
If $A$ is an invertible matrix, then:

- $A^{-1}$ is invertible and $(A^{-1})^{-1} = A$.
- $A^n$ is invertible and $(A^n)^{-1} = (A^{-1})^n$ for $n = 0, 1, \ldots$.
- If $k \neq 0$ then $kA$ is invertible and $(kA)^{-1} = \frac{1}{k} A^{-1}$. 
Matrix Polynomials

Recall that a polynomial is a function of the form

\[ p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n. \]

If \( A \) is an \( m \times m \) matrix then \( p(A) \) is an \( m \times m \) matrix where

\[ p(A) = a_0 I_m + a_1 A + a_2 A^2 + \cdots + a_n A^n. \]
Example

Let \( p(x) = 3x^2 - 2x + 1 \) and \( A = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix} \). Then

\[
p(A) = 3A^2 - 2A + I_2
\]

\[
= 3 \begin{bmatrix} 7 & -10 \\ -15 & 22 \end{bmatrix} - 2 \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

\[
= \begin{bmatrix} 21 & -30 \\ -45 & 66 \end{bmatrix} - \begin{bmatrix} -2 & 4 \\ 6 & -8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

\[
= \begin{bmatrix} 24 & -34 \\ -51 & 75 \end{bmatrix}
\]
Properties of the Transpose

**Theorem**

Let $A$ and $B$ be matrices and $k$ be a scalar, then

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$ and $(A - B)^T = A^T - B^T$
- $(kA)^T = kA^T$
- $(AB)^T = B^T A^T$.

**Remark**: the transpose of the product of any number of matrices is the product of the transposes in reverse order.
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Invertibility of the Transpose

**Theorem**

*If $A$ is an invertible matrix, then $A^T$ is invertible and $(A^T)^{-1} = (A^{-1})^T$.***
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**Proof.**

\[
A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I
\]
Invertibility of the Transpose

**Theorem**

*If A is an invertible matrix, then $A^T$ is invertible and $(A^T)^{-1} = (A^{-1})^T$.***

**Proof.**

\[
A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I
\]

\[
(A^{-1})^T A^T = (AA^{-1})^T = I^T = I
\]
Homework

- Read Section 1.4
- Exercises 1, 2, 4, 5, 6, 8, 11, 13, 14, 16, 17, 21, 22, 29, 32, 39, 46, 49