The Convolution Integral
MATH 365 Ordinary Differential Equations

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Observation: in general $\mathcal{L} \{f(t) g(t)\} \neq \mathcal{L} \{f(t)\} \mathcal{L} \{g(t)\}$;
Observation: in general $\mathcal{L} \{f(t) g(t)\} \neq \mathcal{L} \{f(t)\} \mathcal{L} \{g(t)\}$; however,

if the notion of “product” is modified perhaps the Laplace transform will commute with multiplication.
Generalized Product

Theorem
If \( F(s) = \mathcal{L} \{ f(t) \} \) and \( G(s) = \mathcal{L} \{ g(t) \} \) both exist for \( s > a \geq 0 \), and if we define

\[
h(t) = \int_0^t f(t - \tau) g(\tau) \, d\tau = \int_0^t f(\tau) g(t - \tau) \, d\tau,
\]

then

\[
H(s) = F(s) G(s) = \mathcal{L} \{ h(t) \},
\]

for \( s > a \). The function \( h \) is called the convolution of \( f \) and \( g \). The integrals are known as convolution integrals. The convolution of \( f \) and \( g \) will be denoted \((f \ast g)(t)\).
Well-defined-ness of \((f \ast g)(t)\)

Let \(z = t - \tau\) and \(-dz = d\tau\) then

\[
\int_0^t f(t - \tau) g(\tau) \, d\tau = -\int_t^0 f(z) g(t - z) \, dz
\]

\[
= \int_0^t f(z) g(t - z) \, dz.
\]

Thus the convolution integral is well-defined.
Example

Let $f(t) = \sin t$ and $g(t) = \cos t$ and find

$$(f \ast g)(t)$$
Example

Let \( f(t) = \sin t \) and \( g(t) = \cos t \) and find

\[
(f \ast g)(t) = \int_0^t \sin(t - \tau) \cos \tau \, d\tau
\]

\[
= \int_0^t ((\sin t \cos \tau - \cos t \sin \tau) \cos \tau) \, d\tau
\]

\[
= \int_0^t (\sin t \cos^2 \tau - \cos t \sin \tau \cos \tau) \, d\tau
\]

\[
= \frac{1}{2} \sin t \int_0^t (1 + \cos 2\tau) \, d\tau - \frac{1}{2} \cos t \int_0^t \sin 2\tau \, d\tau
\]

\[
= \frac{t}{2} \sin t + \frac{1}{4} \sin t \sin 2t + \frac{1}{4} \cos t \cos 2t - \frac{1}{4} \cos t
\]

\[
= \frac{t}{2} \sin t
\]
Properties of the Convolution Integral

- \( f \ast g = g \ast f \) (commutativity)
- \( f \ast (g_1 + g_2) = f \ast g_1 + f \ast g_2 \) (distributivity)
- \( (f \ast g) \ast h = f \ast (g \ast h) \) (associativity)
- \( f \ast 0 = 0 \ast f = 0 \)

**Remark:** the convolution possesses many properties similar to multiplication.
Example

Let $f(t)$ be a continuous function and let $g(t) = \delta(t)$ and find

$$(f \ast \delta)(t)$$
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$$(f \ast \delta)(t) = \int_0^t \delta(t - \tau)f(\tau) \, d\tau$$

$$= \int_{-\infty}^\infty \delta(t - \tau)f(\tau) \, d\tau$$

$$= f(t)$$
**Example**

Let $f(t)$ be a continuous function and let $g(t) = \delta(t)$ and find

$$(f \ast \delta)(t) = \int_0^t \delta(t - \tau)f(\tau) \, d\tau$$

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$$= f(t)$$

**Remark:** since convolution is like multiplication, we can think of the Dirac delta function as the identity element.
Justification of Convolution Theorem (1 of 4)

By definition

\[ \mathcal{L} \{ f(t) \} = F(s) = \int_0^\infty e^{-sz} f(z) \, dz \]

\[ \mathcal{L} \{ g(t) \} = G(s) = \int_0^\infty e^{-s\tau} g(\tau) \, d\tau \]

which implies

\[ F(s) G(s) = \int_0^\infty e^{-sz} f(z) \, dz \int_0^\infty e^{-s\tau} g(\tau) \, d\tau \]

\[ = \int_0^\infty e^{-s\tau} g(\tau) \left[ \int_0^\infty e^{-sz} f(z) \, dz \right] \, d\tau \]

\[ = \int_0^\infty g(\tau) \left[ \int_0^\infty e^{-s(z+\tau)} f(z) \, dz \right] \, d\tau. \]
Justification of Convolution Theorem (2 of 4)

Integrate by substitution letting \( t = z + \tau \) and \( dt = dz \), then

\[
F(s) G(s) = \int_0^\infty \int_0^\infty g(\tau) \left[ \int_0^\infty e^{-s(z+\tau)} f(z) \, dz \right] \, d\tau \\
= \int_0^\infty \int_0^\infty g(\tau) \left[ \int_{\tau}^\infty e^{-st} f(t - \tau) \, dt \right] \, d\tau \\
= \int_0^\infty \int_{\tau}^\infty e^{-st} f(t - \tau) g(\tau) \, dt \, d\tau.
\]
The iterated integral is carried out over the region in the $t\tau$-plane described by the set

$$R = \{(t, \tau) : \tau \leq t < \infty \text{ and } 0 < \tau < \infty\}.$$
We can change the order of integration to see that

\[
F(s) \cdot G(s) = \int_0^\infty \int_0^t e^{-st} f(t - \tau) g(\tau) \, d\tau \, dt
\]

\[
= \int_0^\infty e^{-st} \left[ \int_0^t f(t - \tau) g(\tau) \, d\tau \right] \, dt
\]

\[
= \int_0^\infty e^{-st} h(t) \, dt
\]

\[
= \mathcal{L}\{h(t)\}.
\]
Example

Find the Laplace transform of

\[ h(t) = \int_0^t (t - \tau)^2 \cos 2\tau \, d\tau. \]
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\[ h(t) = \int_0^t (t - \tau)^2 \cos 2\tau \, d\tau. \]

\[ \mathcal{L} \{ h(t) \} = \mathcal{L} \{ t^2 \ast \cos 2t \} \]

\[ = \mathcal{L} \{ t^2 \} \mathcal{L} \{ \cos 2t \} \]

\[ = \frac{2}{s^3} \cdot \frac{s}{s^2 + 4} \]

\[ = \frac{2}{s^2(s^2 + 4)} \]
Example

Find the inverse Laplace transform of

\[ \frac{1}{(s + 1)^2 (s^2 + 4)} \]
Let $F(s) = \frac{1}{(s + 1)^2}$ and $G(s) = \frac{1}{s^2 + 4}$, then

$$\mathcal{L}^{-1} \{ F(s) G(s) \} = \int_0^t f(\tau) g(t - \tau) \, d\tau$$

$$= \frac{1}{2} \int_0^t \tau e^{-\tau} \sin(2(t - \tau)) \, d\tau$$

$$= \frac{1}{2} \int_0^t \tau e^{-\tau} \left( \sin 2t \cos 2\tau - \cos 2t \sin 2\tau \right) \, d\tau$$

$$= \frac{1}{2} \sin 2t \int_0^t \tau e^{-\tau} \cos 2\tau \, d\tau$$

$$- \frac{1}{2} \cos 2t \int_0^t \tau e^{-\tau} \sin 2\tau \, d\tau$$

$$= \frac{1}{50} \left[ e^{-t}(4 + 10t) - 4 \cos 2t - 3 \sin 2t \right].$$
Example

Solve the following initial value problem.

\[ y^{(4)} + 5y'' + 4y = g(t) \]
\[ y(0) = 1 \]
\[ y'(0) = 0 \]
\[ y''(0) = 0 \]
\[ y'''(0) = 0 \]
Solution (1 of 2)

\[ G(s) = s^4 Y(s) - s^3 + 5(s^2 Y(s) - s) + 4Y(s) \]

\[ Y(s) = \frac{s}{s^2 + 1} + \frac{s}{(s^2 + 1)(s^2 + 4)} + \frac{G(s)}{(s^2 + 1)(s^2 + 4)} \]

\[ y(t) = \cos t + \frac{1}{2} \int_0^t \sin 2\tau \cos(t - \tau) \, d\tau \]

\[ + \mathcal{L}^{-1} \left\{ \frac{1}{3} \frac{G(s)}{s^2 + 1} - \frac{1}{6} \frac{2G(s)}{s^2 + 4} \right\} \]

\[ = \frac{4}{3} \cos t - \frac{1}{3} \cos 2t \]

\[ + \frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{G(s)}{s^2 + 1} \right\} - \frac{1}{6} \mathcal{L}^{-1} \left\{ \frac{2G(s)}{s^2 + 4} \right\} \]
Solution (2 of 2)

\[ y(t) = \frac{4}{3} \cos t - \frac{1}{3} \cos 2t \]

\[ + \frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{G(s)}{s^2 + 1} \right\} - \frac{1}{6} \mathcal{L}^{-1} \left\{ \frac{2G(s)}{s^2 + 4} \right\} \]

\[ = \frac{4}{3} \cos t - \frac{1}{3} \cos 2t \]

\[ + \frac{1}{3} \int_0^t \sin(t - \tau) g(\tau) \, d\tau - \frac{1}{6} \int_0^t \sin 2(t - \tau) g(\tau) \, d\tau \]

\[ = \frac{4}{3} \cos t - \frac{1}{3} \cos 2t \]

\[ + \frac{1}{3} \int_0^t \left[ \sin(t - \tau) - \frac{1}{2} \sin 2(t - \tau) \right] g(\tau) \, d\tau \]
General Case

Consider the general second-order linear, constant coefficient nonhomogeneous IVP.

\[ ay'' + by' + cy = g(t) \]
\[ y(0) = y_0 \]
\[ y'(0) = y'_0 \]

We can solve this general case using the Laplace transform.

\[
G(s) = a(s^2 Y(s) - sy_0 - y'_0) + b(sY(s) - y_0) + cY(s) \\
= (as^2 + bs + c)Y(s) - (as + b)y_0 - ay'_0 \\
Y(s) = \frac{(as + b)y_0 + ay'_0}{as^2 + bs + c} + \frac{G(s)}{as^2 + bs + c} \\
= \Phi(s) + \Psi(s) \\
y(t) = \mathcal{L}^{-1}\{\Phi(s)\} + \mathcal{L}^{-1}\{\Psi(s)\} \\
= \phi(t) + \psi(t)\]
If \( g(t) = 0 \) then \( \Psi(s) = 0 \) and 

\[
y(t) = \mathcal{L}^{-1} \{\Phi(s)\} = \phi(t) = y_c(t).
\]

We can check that \( y_c(t) \) satisfies the IVP:

\[
ay'' + by' + cy = 0
\]

\[
y(0) = y_0
\]

\[
y'(0) = y'_0
\]
If \( y(0) = 0 \) and \( y'(0) = 0 \) then the original IVP becomes

\[
ay'' + by' + cy = g(t)
\]

\[
y(0) = 0
\]

\[
y'(0) = 0
\]

and \( \Phi(s) = 0 \). Thus \( Y(t) = \psi(t) = \mathcal{L}^{-1}\{\Psi(s)\} \) is the particular solution to the nonhomogeneous equation.
Transfer Function

\[ \psi(s) = \frac{G(s)}{as^2 + bs + c} \]
\[ = G(s)(as^2 + bs + c)^{-1} \]
\[ = G(s)H(s) \]

where \( H(s) = (as^2 + bs + c)^{-1} \) is called the transfer function.

\[ \psi(t) = \mathcal{L}^{-1} \{G(s)H(s)\} = \int_{0}^{t} h(t - \tau) g(\tau) \, d\tau \]

where \( h(t) = \mathcal{L}^{-1} \{H(s)\} \) is called the impulse response of the system.
Homework

- Read Section 6.6
- Exercises: 1–19 odd, 29