Euler Equations
MATH 365 *Ordinary Differential Equations*

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Spring 2015
Euler Equations

An Euler equation is a simple form of second order linear ODE with a regular singular point at $t_0 = 0$,

$$t^2 y'' + \alpha t y' + \beta y = 0$$

where $\alpha$ and $\beta$ are constants.
Let $t = e^z$ then according to the chain rule

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\frac{dy}{dz} = \frac{dy}{dt} \frac{dt}{dz} = y' e^z = t y'
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\frac{d^2 y}{dz^2} - \frac{dy}{dz} = t^2 y''.
\]
By use of the chain rule we have

\[ t^2 y'' = \frac{d^2 y}{dz^2} - \frac{dy}{dz} \]

\[ t \ y' = \frac{dy}{dz}. \]

Substituting into an Euler's equation we obtain

\[ t^2 y'' + \alpha t \ y' + \beta y = 0 \]

\[ \left( \frac{d^2 y}{dz^2} - \frac{dy}{dz} \right) + \alpha \frac{dy}{dz} + \beta y = 0 \]

\[ \frac{d^2 y}{dz^2} + (\alpha - 1) \frac{dy}{dz} + \beta y = 0. \]
Change of Variable (2 of 2)

By use of the chain rule we have

\[ t^2 y'' = \frac{d^2 y}{dz^2} - \frac{dy}{dz} \]
\[ t y' = \frac{dy}{dz}. \]

Substituting into an Euler’s equation we obtain

\[ t^2 y'' + \alpha t y' + \beta y = 0 \]

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\[ \frac{d^2 y}{dz^2} + (\alpha - 1) \frac{dy}{dz} + \beta y = 0. \]

**Note:** the last equation is of the constant coefficient type, and thus we may solve it via the characteristic equation

\[ r^2 + (\alpha - 1)r + \beta = 0. \]
Example

Find the general solution of the following Euler equation.

\[ t^2 y'' + 4t y' + 2y = 0 \]
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Find the general solution of the following Euler equation.

\[ t^2 y'' + 4t \, y' + 2y = 0 \]

After the change of variable \( t = e^z \), we have

\[
\frac{d^2 y}{dz^2} + 3 \frac{dy}{dz} + 2y = 0
\]

\[
r^2 + 3r + 2 = 0
\]

\[
(r + 2)(r + 1) = 0
\]

\[ y(\zeta) = c_1 e^{-2z} + c_2 e^{-z} = c_1 (e^z)^{-2} + c_2 (e^z)^{-1} \]
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\[ y(t) = c_1 t^{-2} + c_2 t^{-1} \]
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Find the general solution of the following Euler equation.

\[ t^2 y'' - 4t y' - 6y = 0 \]
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\frac{d^2 y}{dz^2} - 5 \frac{dy}{dz} - 6y = 0
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\[ r^2 - 5r - 6 = 0 \]

\[ (r - 6)(r + 1) = 0 \]

\[ y(z) = c_1 e^{6z} + c_2 e^{-z} \]
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\[ y(t) = c_1 t^6 + c_2 t^{-1} \]
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\[ y(z) = c_1 e^{2z} + c_2 ze^{2z} \]
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y(z) = c_1 e^{2z} + c_2 z e^{2z}
\]

\[
y(t) = c_1 t^2 + c_2 (\ln t) t^2
\]
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\[ t^2 y'' + t y' + y = 0 \]

After the change of variable \( t = e^z \), we have

\[ \frac{d^2 y}{dz^2} + (0) \frac{dy}{dz} + y = 0 \]

\[ r^2 + 1 = 0 \]
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After the change of variable \( t = e^z \), we have

\[ \frac{d^2 y}{dz^2} + (0) \frac{dy}{dz} + y = 0 \]
\[ r^2 + 1 = 0 \]
\[ y(z) = c_1 \cos z + c_2 \sin z \]
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y(z) = c_1 \cos z + c_2 \sin z
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\[
y(t) = c_1 \cos(\ln t) + c_2 \sin(\ln t)
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Find the general solution of the following nonhomogeneous Euler equation.

\[ t^2 y'' + 7t y' + 5y = t \]
Example (1 of 2)

Find the general solution of the following nonhomogeneous Euler equation.

\[ t^2 y'' + 7t y' + 5y = t \]

First we find the complementary solution:

\[ y_c(t) = c_1 t^{-5} + c_2 t^{-1}. \]
Example (1 of 2)

Find the general solution of the following nonhomogeneous Euler equation.

\[ t^2 y'' + 7t y' + 5y = t \]

First we find the complementary solution:

\[ y_c(t) = c_1 t^{-5} + c_2 t^{-1}. \]

Re-writing the nonhomogeneous equation as

\[ y'' + \frac{7}{t} y' + \frac{5}{t^2} y = \frac{1}{t} \]

we may use the method of Variation of Parameters to find a particular solution.
Example (2 of 2)

\[
W(y_1, y_2)(t) &= 4t^{-7} \\
\mu_1(t) &= -\int_1^t \frac{s^{-1} s^{-1}}{4s^{-7}} \, ds = -\frac{1}{24}(t^6 - 1) \\
\mu_2(t) &= \int_1^t \frac{s^{-1} s^{-5}}{4s^{-7}} \, ds = \frac{1}{8}(t^2 - 1) \\
Y(t) &= -\frac{1}{24}(t^6 - 1)t^{-5} + \frac{1}{8}(t^2 - 1)t^{-1} \\
&= \frac{1}{12}t + \frac{1}{24}t^{-5} + \frac{1}{8}t^{-1}
\]
Example (2 of 2)

\[ W(y_1, y_2)(t) = 4t^{-7} \]

\[ \mu_1(t) = -\int_1^t \frac{s^{-1}s^{-1}}{4s^{-7}} \, ds = -\frac{1}{24} (t^6 - 1) \]

\[ \mu_2(t) = \int_1^t \frac{s^{-1}s^{-5}}{4s^{-7}} \, ds = \frac{1}{8} (t^2 - 1) \]

\[ Y(t) = -\frac{1}{24} (t^6 - 1)t^{-5} + \frac{1}{8} (t^2 - 1)t^{-1} \]

\[ = \frac{1}{12} t + \frac{1}{24} t^{-5} + \frac{1}{8} t^{-1} \]

\[ y(t) = c_1 t^{-5} + c_2 t^{-1} + \frac{1}{12} t \]
Method of Undetermined Coefficients

Luckily we could have used the Method of Undetermined Coefficients to find the particular solution.

Transform the entire nonhomogeneous equation.

\[ t^2 y'' + 7ty' + 5y = t \]
\[ y''(z) + 6y'(z) + 5y(z) = e^z \]

The particular solution should be of the form \( Y(z) = Ae^z \).

\[ Ae^z + 6Ae^z + 5Ae^z = e^z \]
\[ 12A = 1 \quad \implies \quad A = 1/12 \]

\[ Y(z) = \frac{1}{12}e^z \quad \implies \quad Y(t) = \frac{1}{12}t \]

You should not expect this to happen in every example.
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\[ Y(z) = \frac{1}{12}e^z \quad \implies \quad Y(t) = \frac{1}{12}t \]

You should not expect this to happen in every example.
Given an Euler equation \( t^2 y'' + \alpha t \ y' + \beta y = 0 \) the transformed equation is of the form

\[
\frac{d^2 y}{dz^2} + (\alpha - 1) \frac{dy}{dz} + \beta y = 0
\]

where \( t = e^z \).

The transformed equation has characteristic equation

\[
r^2 + (\alpha - 1)r + \beta = 0
\]

which has roots

\[
r_{1,2} = \frac{(1 - \alpha) \pm \sqrt{(\alpha - 1)^2 - 4\beta}}{2}.
\]
If $r_1$ and $r_2 \in \mathbb{R}$ and $r_1 \neq r_2$ then the general solution to Euler’s equation is

$$y(t) = c_1 t^{r_1} + c_2 t^{r_2}.$$ 

If $r_1 = r_2 = r \in \mathbb{R}$ then the general solution to Euler’s equation is

$$y(t) = (c_1 + c_2 (\ln t)) t^r.$$
If \( r_1 = \lambda - \mu i \) and \( r_2 = \lambda + \mu i \) where \( i = \sqrt{-1} \) and \( \mu \neq 0 \) then the general solution to Euler’s equation is

\[
y(t) = t^\lambda \left[ c_1 \cos(\mu \ln t) + c_2 \sin(\mu \ln t) \right].
\]
Homework

- Read Section 5.4
- Exercises: 1–15 odd