Fundamental Solutions of Linear Homogeneous Equations

MATH 365 Ordinary Differential Equations

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Previously we introduced the second order linear homogeneous ODE

\[ y'' + p(t)y' + q(t)y = 0 \]

and showed how to solve the special case

\[ a y'' + b y' + c y = 0 \]

where \( a, b, \) and \( c \) are constants and \( b^2 - 4ac > 0 \).
Second Order Linear Operator

Returning to the more general case:

\[ y'' + p(t)y' + q(t)y = 0 \]

we will assume that \( p(t) \) and \( q(t) \) are continuous on some open interval of the form \((\alpha, \beta)\).

On the set of twice differentiable functions we define the second order linear operator:

\[ L[\phi] = \phi'' + p(t)\phi' + q(t)\phi \]

and thus our second order linear homogeneous ODE can be written as

\[ L[y](t) = y'' + p(t)y' + q(t)y = 0. \]
Theorem
Consider the IVP

\[ y'' + p(t)y' + q(t)y = g(t) \]
\[ y(t_0) = y_0 \]
\[ y'(t_0) = y'_0 \]

where \( p, q, \) and \( g \) are continuous on an open interval \( I \) that contains the point \( t_0 \). Then there exists a unique solution \( y = \phi(t) \) to this IVP and the solution exists for all \( t \in I \).
There are four key ideas in the previous theorem:

1. The IVP has a solution (existence).
2. The IVP has only one solution (uniqueness).
3. The solution is defined throughout the interval $I$ where the coefficient functions are continuous.
4. The solution is at least twice differentiable on interval $I$. 
Example

Consider the IVP

\[(t - 2)y'' - ty' + 3y = e^t\]

\[y(-1) = 1\]

\[y'(-1) = 0\]

Find the largest interval on which there is a unique solution.
Example

Consider the IVP

\[(t - 2)y'' - ty' + 3y = e^t\]
\[y(-1) = 1\]
\[y'(-1) = 0\]

Find the largest interval on which there is a unique solution. Rewrite the ODE in standard form:

\[y'' - \left(\frac{t}{t - 2}\right)y' + \left(\frac{3}{t - 2}\right)y = \frac{e^t}{t - 2}.\]

The largest interval containing the initial condition \(t_0 = -1\) on which the coefficient functions are continuous is \((-\infty, 2)\). Thus the largest interval on which a unique solution to the IVP exists is \((-\infty, 2)\).
Example

Find the largest interval in which the solution to the initial value problem

\[(t^2 - 3t)y'' + ty' - (t + 3)y = 0\]

\[y(1) = 2\]
\[y'(1) = 1\]

is certain to exist.
Linear Combination of Solutions

Theorem (Principle of Superposition)

If $y_1$ and $y_2$ are two solutions of the ODE

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

then the linear combination $c_1y_1 + c_2y_2$ is also a solution for any values of the constants $c_1$ and $c_2$. 
Linear Combination of Solutions

Theorem (Principle of Superposition)

If $y_1$ and $y_2$ are two solutions of the ODE

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

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Proof.

$$L[c_1 y_1 + c_2 y_2] = c_1 L[y_1] + c_2 L[y_2]$$

$$= c_1 (y_1'' + p(t)y_1' + q(t)y_1)$$

$$= 0$$

$$+ c_2 (y_2'' + p(t)y_2' + q(t)y_2) = 0$$

$$= 0$$
Initial Conditions

Given two solutions to the ODE (say $y_1$ and $y_2$) we can construct an infinite number of solutions to the ODE of the form $c_1y_1 + c_2y_2$. 
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**Question:** can any solution to the IVP be expressed as a linear combination of $y_1$ and $y_2$?
Cramer’s Rule

To satisfy the initial conditions we must be able to solve the following equations for $c_1$ and $c_2$.

\[
\begin{align*}
c_1 y_1(t_0) + c_2 y_2(t_0) &= y_0 \\
c_1 y'_1(t_0) + c_2 y'_2(t_0) &= y'_0
\end{align*}
\]
Cramer’s Rule

To satisfy the initial conditions we must be able to solve the following equations for \( c_1 \) and \( c_2 \).

\[
\begin{align*}
    c_1 y_1(t_0) + c_2 y_2(t_0) &= y_0 \\
    c_1 y_1'(t_0) + c_2 y_2'(t_0) &= y_1'
\end{align*}
\]

Applying Cramer’s Rule we have

\[
\begin{align*}
    c_1 &= \frac{y_0 y_2'(t_0) - y_0' y_2(t_0)}{y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0)} \\
    c_2 &= \frac{-y_0 y_1'(t_0) + y_0' y_1(t_0)}{y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0)}
\end{align*}
\]

provided the denominators are non-zero.
Wronskian Determinant

Definition
The **Wronskian determinant** or **Wronskian** of the solutions $y_1$ and $y_2$ is

$$W = W(y_1, y_2)(t_0)$$

$$= \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}$$

$$= y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0).$$
Solutions to the IVP

Theorem
Consider the IVP

\[ y'' + p(t)y' + q(t)y = 0 \]
\[ y(t_0) = y_0 \]
\[ y'(t_0) = y'_0 \]

Suppose that \( y_1 \) and \( y_2 \) are two solutions to the ODE and that the Wronskian

\[ W(y_1, y_2)(t) = y_1(t)y'_2(t) - y_1'(t)y_2(t) \]

is not zero at the point \( t_0 \), then there is a choice of constants \( c_1 \) and \( c_2 \) for which \( y = c_1y_1 + c_2y_2 \) satisfies the IVP.
Example

Consider the IVP

\[(1 - t \cot t)y'' - ty' + y = 0\]
\[y(\pi/2) = 1\]
\[y'(\pi/2) = 0\]

Verify that \(y_1(t) = t\) and that \(y_2(t) = \sin t\) are solutions to the ODE. Calculate the Wronskian and find a solution to the IVP.
General Solution

Theorem

If $y_1$ and $y_2$ are two solutions of the ODE

$$y'' + p(t)y' + q(t)y = 0$$

and if there is a point $t_0$ where the Wronskian of $y_1$ and $y_2$ is non-zero, then the family of solutions

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

with arbitrary constants $c_1$ and $c_2$ includes every solution of the ODE.

$y(t) = c_1 y_1(t) + c_2 y_2(t)$ is called the general solution of the ODE and $\{y_1(t), y_2(t)\}$ are said to form a fundamental set of solutions to the ODE.
Example

Find a fundamental set of solutions and the general solution to the ODE

\[ y'' + 4y' + 3y = 0. \]
Final Result

Sometimes finding the fundamental set of solutions through direct computation is difficult.

**Theorem**

*Consider the ODE*

\[ y'' + p(t)y' + q(t)y = 0 \]

*whose coefficients p and q are continuous on some open interval I. Let \( t_0 \in I \) and let \( y_1(t) \) be the solution of the ODE satisfying*

\[ y_1(t_0) = 1, \quad y_1'(t_0) = 0 \]

*while \( y_2(t) \) is the solution of the ODE satisfying*

\[ y_2(t_0) = 0, \quad y_2'(t_0) = 1. \]

*Then \( \{y_1(t), y_2(t)\} \) is a fundamental set of solutions to the ODE.*
Example

Find a fundamental set of solutions to the ODE

\[ y'' + 5y' + 4y = 0 \]

at \( t_0 = 1 \).
Homework

- Read Section 3.2
- Exercises: 1–25 odd