Millersville University **Name** **Answer Key**
Department of Mathematics
MATH 365, *Ordinary Differential Equations*, Homework 01
August 27, 2008

Please answer the following questions covering topics you learned in calculus courses. Answers without justifying work will receive no credit. Partial credit will be given as appropriate, do not leave any problem blank. Each problem is worth 10 points. Your completed assignment is due at class time on Friday, August 29, 2008.

1. Please evaluate the following indefinite integrals.

   (a) $\int x \sin(x^2)\, dx$

   Using integration by substitution and letting

$$\begin{aligned} u &= x^2 \\ \frac{1}{2}\, du &= x\, dx \end{aligned}$$

   we obtain

$$\begin{aligned} \int x \sin(x^2)\, dx &= \frac{1}{2} \int \sin u\, du \\ &= -\frac{1}{2} \cos u + C \\ &= -\frac{1}{2} \cos(x^2) + C. \end{aligned}$$

   (b) $\int \frac{\ln(\ln x)}{x}\, dx$

   Using integration by substitution and letting

$$\begin{aligned} w &= \ln x \\ dw &= \frac{1}{x}\, dx \end{aligned}$$

   we obtain

$$\int \frac{\ln(\ln x)}{x}\, dx = \int \ln w\, dw.$$

   Now we will apply integration by parts with

$$\begin{aligned} u &= \ln w & \qquad v &= w \\ du &= \tfrac{1}{w}\, dw & \qquad dv &= dw. \end{aligned}$$

Then we have

$$\begin{aligned}
\int \ln w \, dw &= w \ln w - \int w \left(\frac{1}{w} \right) \, dw \\
&= w \ln w - \int 1 \, dw \\
&= w \ln w - w + C \\
\int \frac{\ln(\ln x)}{x} \, dx &= (\ln x) \ln(\ln x) - \ln x + C.
\end{aligned}$$

(c) $\displaystyle\int \frac{1}{x(K-x)} \, dx$, where $0 < x < K$.

Since the integrand is a rational function we will use partial fraction decomposition to write

$$\begin{aligned}
\frac{1}{x(K-x)} &= \frac{A}{x} + \frac{B}{K-x} \\
&= \frac{A(K-x) + Bx}{x(K-x)} \\
1 &= A(K-x) + Bx.
\end{aligned}$$

If $x = 0$ then we see that $A = 1/K$. If $x = K$ then we have $B = 1/K$. Therefore

$$\begin{aligned}
\int \frac{1}{x(K-x)} \, dx &= \int \left(\frac{1/K}{x} + \frac{1/K}{K-x} \right) \, dx \\
&= \frac{1}{K} \int \left(\frac{1}{x} + \frac{1}{K-x} \right) \, dx \\
&= \frac{1}{K} \left(\ln|x| - \ln|K-x| \right) + C \\
&= \frac{1}{K} \ln \left(\frac{x}{K-x} \right) + C
\end{aligned}$$

since $0 < x < K$.

(d) $\displaystyle\int 3xe^{x/4} + 2e^{x/4} \sin 2x \, dx$

We will break this integral into two integrals and work on each part separately for simplicity.

$$\int 3xe^{x/4} + 2e^{x/4} \sin 2x \, dx = \int 3xe^{x/4} \, dx + \int 2e^{x/4} \sin 2x \, dx$$

Taking the first integral on the right hand side we will use integration by parts with

$$\begin{aligned}
u &= 3x & v &= 4e^{x/4} \\
du &= 3 \, dx & dv &= e^{x/4} \, dx.
\end{aligned}$$

Then

$$\begin{aligned}
\int 3xe^{x/4}\,dx &= (3x)4e^{x/4} - \int 4e^{x/4}3\,dx \\
&= 12xe^{x/4} - 12\int e^{x/4}\,dx \\
&= 12xe^{x/4} - 48e^{x/4} + C_1.
\end{aligned}$$

Taking the second integral on the right hand side we will use integration by parts with

$$\begin{aligned}
u &= e^{x/4} & v &= -\cos 2x \\
du &= \tfrac{1}{4}e^{x/4}\,dx & dv &= 2\sin 2x\,dx.
\end{aligned}$$

Then

$$\begin{aligned}
\int 2e^{x/4}\sin 2x\,dx &= -e^{x/4}\cos 2x - \int \frac{1}{4}e^{x/4}(-\cos 2x)\,dx \\
&= -e^{x/4}\cos 2x + \int \frac{1}{4}e^{x/4}\cos 2x\,dx.
\end{aligned}$$

Once again we will use integration by parts with now

$$\begin{aligned}
u &= \tfrac{1}{4}e^{x/4} & v &= \tfrac{1}{2}\sin 2x \\
du &= \tfrac{1}{16}e^{x/4}\,dx & dv &= \cos 2x\,dx.
\end{aligned}$$

Then

$$\begin{aligned}
\int 2e^{x/4}\sin 2x\,dx &= -e^{x/4}\cos 2x + \int \frac{1}{4}e^{x/4}\cos 2x\,dx \\
&= -e^{x/4}\cos 2x + \frac{1}{4}e^{x/4}\frac{1}{2}\sin 2x - \int \frac{1}{16}e^{x/4}\frac{1}{2}\sin 2x\,dx \\
&= -e^{x/4}\cos 2x + \frac{1}{8}e^{x/4}\sin 2x - \frac{1}{32}\int e^{x/4}\sin 2x\,dx \\
&= -e^{x/4}\cos 2x + \frac{1}{8}e^{x/4}\sin 2x - \frac{1}{64}\int 2e^{x/4}\sin 2x\,dx \\
\int 2e^{x/4}\sin 2x\,dx + \frac{1}{64}\int 2e^{x/4}\sin 2x\,dx &= -e^{x/4}\cos 2x + \frac{1}{8}e^{x/4}\sin 2x \\
\frac{65}{64}\int 2e^{x/4}\sin 2x\,dx &= \\
\int 2e^{x/4}\sin 2x\,dx &= \frac{64}{65}\left(-e^{x/4}\cos 2x + \frac{1}{8}e^{x/4}\sin 2x\right) + C_2.
\end{aligned}$$

So finally

$$\begin{aligned}
\int 3xe^{x/4} + 2e^{x/4}\sin 2x\,dx &= \int 3xe^{x/4}\,dx + \int 2e^{x/4}\sin 2x\,dx \\
&= 12xe^{x/4} - 48e^{x/4} + C_1 + \frac{64}{65}\left(-e^{x/4}\cos 2x + \frac{1}{8}e^{x/4}\sin 2x\right) + C_2 \\
&= 12xe^{x/4} - 48e^{x/4} - \frac{64}{65}e^{x/4}\cos 2x + \frac{8}{65}e^{x/4}\sin 2x + C.
\end{aligned}$$

(e) $\displaystyle\int \frac{(\arctan x)^5}{1+x^2}\,dx$

We will integrate by substitution letting

$$\begin{aligned} u &= \arctan x \\ du &= \frac{1}{1+x^2}\,dx \end{aligned}$$

and obtain

$$\begin{aligned} \int \frac{(\arctan x)^5}{1+x^2}\,dx &= \int u^5\,du \\ &= \frac{1}{6}u^6 + C \\ &= \frac{1}{6}(\arctan x)^6 + C. \end{aligned}$$

2. Assuming that y is implicitly a function of x, find the following derivative:

$$\frac{d}{dx}\left(y\sin x + x^2 e^y - y\right).$$

$$\begin{aligned} \frac{d}{dx}\left(y\sin x + x^2 e^y - y\right) &= \frac{dy}{dx}\sin x + y\cos x + 2xe^y + x^2 e^y \frac{dy}{dx} - \frac{dy}{dx} \\ &= y\cos x + 2xe^y + (\sin x + x^2 e^y - 1)\frac{dy}{dx} \end{aligned}$$

3. Use a sum or difference of angles formula to write the sum of trigonometric expressions

$$\frac{1}{2}\cos(2t) + \frac{\sqrt{3}}{2}\sin(2t)$$

in the form $A\cos(\omega t + \gamma)$. Clearly specify A, ω, and γ.

By the sum of angles formula for the cosine

$$\begin{aligned} A\cos(\omega t + \gamma) &= A(\cos(\omega t)\cos\gamma - \sin(\omega t)\sin\gamma) \\ &= A\cos\gamma\cos(\omega t) - A\sin\gamma\sin(\omega t). \end{aligned}$$

We would like to pick A, ω, and γ so that

$$\frac{1}{2}\cos(2t) + \frac{\sqrt{3}}{2}\sin(2t) = A\cos\gamma\cos(\omega t) - A\sin\gamma\sin(\omega t).$$

We can let $\omega = 2$, $A = 1$, and $\gamma = -\pi/3$ and see that

$$\frac{1}{2}\cos(2t) + \frac{\sqrt{3}}{2}\sin(2t) = \cos(2t - \pi/3).$$

4. Determine the radius of convergence of the Maclaurin series

$$\sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}$$

We can re-write the series as

$$\sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$$

which is a geometric series. This series converges if and only if $\left|\frac{x}{2}\right| < 1$, or equivalently $|x| < 2$. This implies the radius of convergence of the series is $\rho = 2$.

5. Find the Taylor series expansion for $\ln(1 + x)$ centered at $c = 0$.

Note that $\frac{d}{dx} \ln(1 + x) = \frac{1}{1 + x}$ and that

$$\begin{aligned} \frac{1}{1+x} &= \frac{1}{1-(-x)} \\ &= \sum_{n=0}^{\infty} (-x)^n \\ &= \sum_{n=0}^{\infty} (-1)^n x^n, \end{aligned}$$

if $|x| < 1$. Integrating this power series we have

$$\ln(1 + x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n + 1} x^{n+1}.$$

6. If the function $f(x)$ is defined as

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

and has a positive radius of convergence, then find the power series for $f''(x)$.

A power series with a positive radius of convergence may be differentiated term-by-term inside its interval of convergence in order to find the power series for the derivative of the original power series. The radius of convergence of the derivative series will be at least as large as the radius of convergence of the original series.

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n x^n \\ f'(x) &= \frac{d}{dx} \left(\sum_{n=0}^{\infty} a_n x^n \right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \left(\frac{d}{dx} a_n x^n \right) \\
&= \sum_{n=0}^{\infty} a_n \left(\frac{d}{dx} x^n \right) \\
&= \sum_{n=0}^{\infty} a_n \left(n x^{n-1} \right) \\
&= \sum_{n=0}^{\infty} n a_n x^{n-1} \\
&= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
f''(x) &= \frac{d}{dx} \left(\sum_{n=1}^{\infty} n a_n x^{n-1} \right) \\
&= \sum_{n=1}^{\infty} \left(\frac{d}{dx} n a_n x^{n-1} \right) \\
&= \sum_{n=1}^{\infty} n a_n \left(\frac{d}{dx} x^{n-1} \right) \\
&= \sum_{n=1}^{\infty} n a_n \left((n-1) x^{n-2} \right) \\
&= \sum_{n=1}^{\infty} n(n-1) a_n x^{n-2} \\
&= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}
\end{aligned}
$$