Millersville University **Name** **Answer Key**
Department of Mathematics
MATH 365, *Ordinary Differential Equations*, Homework 02
September 3, 2008

Find the solutions to the following ordinary differential equations and initial value problems. Answers without justifying work will receive no credit. Partial credit will be given as appropriate, do not leave any problem blank. Each problem is worth 10 points. Your completed assignment is due at class time on Friday, September 5, 2008.

1. $\dfrac{dy}{dt} = \dfrac{(y-1)(t-2)(y+3)}{(t-1)(y-2)(t+3)}$

   This ODE is separable and can be written as

$$\begin{aligned}\frac{dy}{dt} &= \frac{(y-1)(t-2)(y+3)}{(t-1)(y-2)(t+3)}\\ \frac{y-2}{(y-1)(y+3)}\,dy &= \frac{t-2}{(t-1)(t+3)}\,dt.\end{aligned}$$

   We can use partial fraction decomposition to write

$$\frac{y-2}{(y-1)(y+3)} = \frac{-1/4}{y-1} + \frac{5/4}{y+3}$$

   and thus

$$\begin{aligned}\frac{y-2}{(y-1)(y+3)}\,dy &= \frac{t-2}{(t-1)(t+3)}\,dt\\ \left(\frac{-1/4}{y-1} + \frac{5/4}{y+3}\right)dy &= \left(\frac{-1/4}{t-1} + \frac{5/4}{t+3}\right)dt\\ \int\left(\frac{5}{y+3} - \frac{1}{y-1}\right)dy &= \int\left(\frac{5}{t+3} - \frac{1}{t-1}\right)dt\\ 5\ln|y+3| - \ln|y-1| &= 5\ln|t+3| - \ln|t-1| + C\\ \ln\left|\frac{(y+3)^5}{y-1}\right| &= \ln\left|\frac{(t+3)^5}{t-1}\right| + C\\ \left|\frac{(y+3)^5}{y-1}\right| &= C\left|\frac{(t+3)^5}{t-1}\right|.\end{aligned}$$

2. $\dfrac{dy}{dt} + \cot t \, y = \cos t$

This is a first order linear ODE. We will need an integrating factor of the form:

$$\begin{aligned} \mu(t) &= e^{\int \cot t \, dt} \\ &= e^{\int \frac{\cos t}{\sin t} \, dt} \\ &= e^{\ln|\sin t|} \\ &= \sin t \end{aligned}$$

provided $0 < t < \pi$. Multiplying both sides of the ODE by the integrating factor produces

$$\begin{aligned} \sin t \left(\frac{dy}{dt} + \cot t \, y \right) &= \sin t \, (\cos t) \\ \frac{d}{dt} (\sin t \, y) &= \cos t \sin t \\ \sin t \, y(t) &= \int \cos t \sin t \, dt \\ &= \frac{1}{2} \sin^2 t + C \\ y(t) &= \frac{1}{2} \sin t + C \csc t \end{aligned}$$

3. $\dfrac{dy}{dt} = -\dfrac{3t+ty^2}{2y+t^2y}$ and $y(\pi/2) = 0$

This ODE is separable.

$$\begin{aligned}
\frac{dy}{dt} &= -\frac{3t+ty^2}{2y+t^2y} \\
&= -\frac{t(3+y^2)}{y(2+t^2)} \\
\frac{y}{3+y^2}\,dy &= -\frac{t}{2+t^2}\,dt \\
\int_0^y \frac{s}{3+s^2}\,ds &= -\int_{\pi/2}^t \frac{s}{2+s^2}\,ds \\
\frac{1}{2}\ln|3+s^2|\Big|_0^y &= -\frac{1}{2}\ln|2+s^2|\Big|_{\pi/2}^t \\
\frac{1}{2}\ln(3+y^2) - \frac{1}{2}\ln 3 &= -\frac{1}{2}\ln(2+t^2) + \frac{1}{2}\ln\left(2+\frac{\pi^2}{4}\right) \\
\ln(3+y^2) - \ln 3 &= -\ln(2+t^2) + \ln\left(2+\frac{\pi^2}{4}\right) \\
\ln(3+y^2) &= -\ln(2+t^2) + \ln 3 + \ln\left(2+\frac{\pi^2}{4}\right) \\
&= -\ln(2+t^2) + \ln\left(6+\frac{3\pi^2}{4}\right) \\
&= \ln\left(\frac{6+\frac{3\pi^2}{4}}{2+t^2}\right) \\
&= \ln\left(\frac{24+3\pi^2}{8+4t^2}\right) \\
3+y^2 &= \frac{24+3\pi^2}{8+4t^2} \\
y^2 &= -3+\frac{24+3\pi^2}{8+4t^2} \\
&= \frac{3\pi^2-12t^2}{8+4t^2}
\end{aligned}$$

4. $\dfrac{dy}{dt} = t - \dfrac{y}{3t}$ and $y(1) = 1$

If we rearrange terms we see that this ODE is first order linear.

$$\frac{dy}{dt} + \frac{1}{3t}y = t$$

An integrating factor for this ODE is

$$\begin{aligned} \mu(t) &= e^{\int \frac{1}{3t}\,dt} \\ &= e^{\frac{1}{3}\ln|t|} \\ &= t^{1/3} \end{aligned}$$

assuming $t > 0$. Multiplying both sides of the ODE by the integrating factor produces

$$\begin{aligned} t^{1/3}\left(\frac{dy}{dt} + \frac{1}{3t}y\right) &= t^{1/3}(t) \\ \frac{d}{dt}\left(t^{1/3}y\right) &= t^{4/3} \\ t^{1/3}y(t) &= \frac{3}{7}t^{7/3} + C \\ y(t) &= \frac{3}{7}t^2 + \frac{C}{t^{1/3}} \end{aligned}$$

Since $y(1) = 1$ then

$$1 = \frac{3}{7} + C \qquad \Longrightarrow \qquad C = \frac{4}{7}$$

and thus the solution to the IVP is

$$y(t) = \frac{3}{7}t^2 + \frac{4}{7t^{1/3}}.$$

5. $\dfrac{dy}{dt} = \dfrac{4y^2 - t^4}{4ty}$ (*Hint*: change the dependent variable to $v = y/t$. Use this to find the solution to the original equation.)

If $v = y/t$ then $y = vt$ and

$$\frac{dy}{dt} = \frac{dv}{dt}t + v$$

according to the product rule for derivatives. Substituting into the ODE we obtain

$$\begin{aligned}
\frac{dy}{dt} &= \frac{4y^2 - t^4}{4ty} \\
\frac{dv}{dt}t + v &= \frac{4(vt)^2 - t^4}{4t(vt)} \\
&= \frac{4v^2t^2 - t^4}{4vt^2} \\
&= \frac{4v^2 - t^2}{4v} \\
\frac{dv}{dt}t &= \frac{4v^2 - t^2}{4v} - v \\
&= \frac{4v^2 - t^2 - 4v^2}{4v} \\
&= -\frac{t^2}{4v} \\
\frac{dv}{dt} &= -\frac{t}{4v} \\
2v\,dv &= -\frac{1}{2}t\,dt \\
v^2 &= -\frac{1}{4}t^2 + C \\
\left(\frac{y}{t}\right)^2 &= -\frac{1}{4}t^2 + C \\
y^2 &= -\frac{1}{4}t^4 + Ct^2.
\end{aligned}$$

6. $\dfrac{dy}{dt} + 2y = y \ln y$ (*Hint*: change the dependent variable to $v = \ln y$. Use this to find the solution to the original equation.)

If $v = \ln y$ then $e^v = y$ and

$$\frac{dy}{dt} = e^v \frac{dv}{dt}$$

according to the chain rule for derivatives. Substituting into the ODE gives us

$$\begin{aligned} \frac{dy}{dt} + 2y &= y \ln y \\ e^v \frac{dv}{dt} + 2e^v &= e^v \ln(e^v) \\ \frac{dv}{dt} + 2 &= \ln(e^v) \\ \frac{dv}{dt} + 2 &= v \\ \frac{dv}{dt} - v &= -2. \end{aligned}$$

This is a first order linear ODE which we will solve using the integrating factor

$$\mu(t) = e^{\int -1 \, dt} = e^{-t}.$$

Multiplying both sides of the ODE by the integrating factor yields

$$\begin{aligned} e^{-t}\left(\frac{dv}{dt} - v\right) &= -2e^{-t} \\ \frac{d}{dt}\left(e^{-t}v\right) &= -2e^{-t} \\ e^{-t}v(t) &= 2e^{-t} + C \\ v(t) &= 2 + Ce^t \\ \ln y(t) &= 2 + Ce^t \\ y(t) &= e^{2+Ce^t}. \end{aligned}$$