Millersville University **Name** <u>**Answer Key**</u>
Department of Mathematics
MATH 365, *Ordinary Differential Equations*, Homework 03
September 10, 2008

Answer the following questions by solving the appropriate first order linear differential equations. Answers without justifying work will receive no credit. Partial credit will be given as appropriate, do not leave any problem blank. Each problem is worth 10 points. Your completed assignment is due at class time on Friday, September 12, 2008.

1. When the drug Theophylline is administered for asthma, a concentration in the blood below 5 mg/L has little effect, but undesirable side effects appear when the concentration exceeds 20 mg/L. Suppose a dose corresponding to 14 mg/L of blood is administered initially. The concentration satisfies the differential equation

$$\frac{dC}{dt} = -\frac{1}{6}C$$

where t is measured in hours.

(a) Find the concentration at time t.

We are solving the IVP

$$\begin{aligned} \frac{dC}{dt} &= -\frac{1}{6}C \\ C(0) &= 14. \end{aligned}$$

This is the same form of differential equation as encountered in mathematical models of exponential decay. The solution is

$$C(t) = 14e^{-t/6}.$$

(b) How long after the initial injection will a second injection be needed to prevent the drug from becoming ineffective?

Since the drug becomes ineffective at a concentration of 5 mg/L in the blood we can solve the following equation.

$$\begin{aligned} C(t^*) &= 5 \\ 14e^{-t^*/6} &= 5 \\ e^{-t^*/6} &= \frac{5}{14} \\ -\frac{t^*}{6} &= \ln\left(\frac{5}{14}\right) \\ \frac{t^*}{6} &= \ln\left(\frac{14}{5}\right) \\ t^* &= 6\ln\left(\frac{14}{5}\right) \\ &\approx 6.17772 \quad \text{hours} \end{aligned}$$

(c) If each injection increases the concentration of the drug by 14 mg/L, what is the minimum safe time after the second injection that a third injection may be given?

If we now consider the IVP

$$\begin{aligned} \frac{dC}{dt} &= -\frac{1}{6}C \\ C(t^*) &= 14+5, \end{aligned}$$

the solution for $t > t^*$ is

$$C(t) = 19e^{-(t-t^*)/6}.$$

It will not be safe to administer another injection of concentration 14 mg/L until this function reaches a concentration of 6 mg/L (since undesireable side effects appear at concentrations above 20 mg/L).

$$\begin{aligned} C(t) &= 6 \\ 19e^{-(t-t^*)/6} &= 6 \\ e^{-(t-t^*)/6} &= \frac{6}{19} \\ -\frac{(t-t^*)}{6} &= \ln\left(\frac{6}{19}\right) \\ \frac{(t-t^*)}{6} &= \ln\left(\frac{19}{6}\right) \\ t-t^* &= 6\ln\left(\frac{19}{6}\right) \\ &\approx 6.91608 \quad \text{hours} \end{aligned}$$

2. Let functions $p(t)$, $s(t)$, and $d(t)$ denote respectively the price, supply, and demand of a product at time t. Allen's Speculative Model in economics assumes that s and d are linear combinations of $p(t)$ and $p'(t)$.

$$\begin{aligned} s(t) &= a_1p(t) + a_2p'(t) + a_3 \qquad (1) \\ d(t) &= b_1p(t) + b_2p'(t) + b_3 \qquad (2) \end{aligned}$$

where the a_i's and b_i's are constants. The economic principle of supply and demand which guarantees a state of market equilibrium is

$$s(t) = d(t). \qquad (3)$$

(a) By combining Eqs. (1), (2), and (3) find a single linear differential equation for $p(t)$.

$$\begin{aligned} s(t) &= d(t) \qquad \text{(Eq. (3))} \\ \text{(Eq. (1))} \quad a_1p(t) + a_2p'(t) + a_3 &= b_1p(t) + b_2p'(t) + b_3 \quad \text{(Eq. (2))} \\ (a_2 - b_2)p'(t) &= (b_1 - a_1)p(t) + (b_3 - a_3) \\ p'(t) + \left(\frac{b_1 - a_1}{b_2 - a_2}\right) p(t) &= -\left(\frac{b_3 - a_3}{b_2 - a_2}\right) \end{aligned}$$

(b) Assuming that $a_1 \neq b_1$, $a_2 \neq b_2$, and $a_3 \neq b_3$, solve the differential equation subject to the initial condition $p(0) = p_0$.

The ODE just found is of the first order linear type and thus we may solve it by making use of an integrating factor.

$$\mu(t) = e^{\frac{(b_1-a_1)t}{(b_2-a_2)}}$$

$$\begin{aligned} \mu(t)\left[p'(t) + \left(\frac{b_1 - a_1}{b_2 - a_2}\right) p(t)\right] &= -\mu(t)\left(\frac{b_3 - a_3}{b_2 - a_2}\right) \\ \frac{d}{dt}\left[p(t)e^{\frac{(b_1-a_1)t}{(b_2-a_2)}}\right] &= -\left(\frac{b_3 - a_3}{b_2 - a_2}\right) e^{\frac{(b_1-a_1)t}{(b_2-a_2)}} \\ p(t)e^{\frac{(b_1-a_1)t}{(b_2-a_2)}} - p(0) &= -\int_0^t \left(\frac{b_3 - a_3}{b_2 - a_2}\right) e^{\frac{(b_1-a_1)s}{(b_2-a_2)}}\, ds \\ &= -\frac{(b_3 - a_3)}{(b_1 - a_1)} e^{\frac{(b_1-a_1)s}{(b_2-a_2)}}\bigg|_0^t \\ &= -\frac{(b_3 - a_3)}{(b_1 - a_1)} e^{\frac{(b_1-a_1)t}{(b_2-a_2)}} + \frac{(b_3 - a_3)}{(b_1 - a_1)} \\ p(t)e^{\frac{(b_1-a_1)t}{(b_2-a_2)}} &= p(0) + \frac{(b_3 - a_3)}{(b_1 - a_1)} - \frac{(b_3 - a_3)}{(b_1 - a_1)} e^{\frac{(b_1-a_1)t}{(b_2-a_2)}} \\ p(t) &= \left(p(0) + \frac{(b_3 - a_3)}{(b_1 - a_1)}\right) e^{-\frac{(b_1-a_1)t}{(b_2-a_2)}} - \frac{(b_3 - a_3)}{(b_1 - a_1)} \end{aligned}$$

(c) Suppose that

$$\begin{aligned} s(t) &= p(t) + 4p'(t) + 30 \\ d(t) &= -2p(t) + 3p'(t) + 48 \end{aligned}$$

and $p(0) = 10$. What happens to the price as t increases?
In this case the solution has the form

$$\begin{aligned} p(t) &= \left(p(0) + \frac{(b_3 - a_3)}{(b_1 - a_1)}\right) e^{-\frac{(b_1 - a_1)t}{(b_2 - a_2)}} - \frac{(b_3 - a_3)}{(b_1 - a_1)} \\ &= \left(10 + \frac{(48 - 30)}{(-2 - 1)}\right) e^{-\frac{(-2-1)t}{(3-4)}} - \frac{(48 - 30)}{(-2 - 1)} \\ &= \left(10 + \frac{18}{-3}\right) e^{-\frac{-3t}{-1}} - \frac{18}{-3} \\ &= (10 - 6)\, e^{-3t} + 6 \\ &= 4e^{-3t} + 6. \end{aligned}$$

Therefore we can see that

$$\lim_{t\to\infty} p(t) = 6.$$

(d) Suppose that $\dfrac{b_1 - a_1}{b_2 - a_2} > 0$. What happens to the price as t increases?

$$\begin{aligned} \lim_{t\to\infty} p(t) &= \lim_{t\to\infty} \left(p(0) + \frac{(b_3 - a_3)}{(b_1 - a_1)}\right) e^{-\frac{(b_1 - a_1)t}{(b_2 - a_2)}} - \frac{(b_3 - a_3)}{(b_1 - a_1)} \\ &= -\frac{(b_3 - a_3)}{(b_1 - a_1)} \end{aligned}$$

(e) Suppose that $\dfrac{b_1 - a_1}{b_2 - a_2} < 0$. What happens to the price as t increases?

$$\begin{aligned} \lim_{t\to\infty} p(t) &= \lim_{t\to\infty} \left(p(0) + \frac{(b_3 - a_3)}{(b_1 - a_1)}\right) e^{-\frac{(b_1 - a_1)t}{(b_2 - a_2)}} - \frac{(b_3 - a_3)}{(b_1 - a_1)} \\ &= \infty \end{aligned}$$