Millersville University **Name** **Answer Key**
Department of Mathematics
MATH 365, *Ordinary Differential Equations*, Homework 04
September 17, 2008

Answer the following questions by solving the appropriate first order differential equations. Answers without justifying work will receive no credit. Partial credit will be given as appropriate, do not leave any problem blank. Each problem is worth 10 points. Your completed assignment is due at class time on Friday, September 19, 2008.

1. Show the following equation is exact and find its general solution.

$$\left(\frac{y}{(t+y)^2}-1\right)dt+\left(1-\frac{t}{(t+y)^2}\right)dy=0$$

We see that

$$\begin{aligned}\frac{\partial}{\partial y}\left(\frac{y}{(t+y)^2}-1\right) &= \frac{(t+y)^2-2y(t+y)}{(t+y)^4}\\ &= \frac{t-y}{(t+y)^3}\\ &= -\frac{(t+y)^2-2t(t+y)}{(t+y)^4}\\ &= \frac{\partial}{\partial t}\left(1-\frac{t}{(t+y)^2}\right)\end{aligned}$$

and thus the equation is exact.

If we assume that y is an implicit function of t, then

$$\begin{aligned}\psi(t,y) &= \int\left(\frac{y}{(t+y)^2}-1\right)dt\\ &= -\frac{y}{t+y}-t+h(y)\end{aligned}$$

where h is an arbitrary function of y.

$$\begin{aligned}\frac{\partial}{\partial y}\psi(t,y) &= \frac{\partial}{\partial y}\left(-\frac{y}{t+y}-t+h(y)\right)\\ &= \frac{-(t+y)+y}{(t+y)^2}+h'(y)\\ &= -\frac{t}{(t+y)^2}+h'(y)\\ &= 1-\frac{t}{(t+y)^2}\end{aligned}$$

which implies $h'(y) = 1$ and that $h(y) = y$. Thus the implicit form of the solution to the exact equation is

$$y - \frac{y}{t+y} - t = C$$

where C is an arbitrary constant.

2. Determine the most general form of the function $N(t, y)$ such that

$$(y \sin t + t^2 y - t \sec y)\, dt + N(t, y)\, dy = 0$$

is exact.

If we assume the equation is exact and that y is an implicit function of t then

$$\begin{aligned} \psi(t, y) &= \int (y \sin t + t^2 y - t \sec y)\, dt \\ &= -y \cos t + \frac{1}{3} t^3 y - \frac{1}{2} t^2 \sec y + H(y) \end{aligned}$$

where H is an arbitrary function of y. Then

$$\begin{aligned} \frac{\partial}{\partial y} \psi(t, y) &= \frac{\partial}{\partial y} \left(-y \cos t + \frac{1}{3} t^3 y - \frac{1}{2} t^2 \sec y + H(y) \right) \\ N(t, y) &= -\cos t + \frac{1}{3} t^3 - \frac{1}{2} t^2 \sec y \tan y + h(y) \end{aligned}$$

where $h(y) = H'(y)$.

3. Show that in general the ODE

$$y f(t\,y)\,dt + t g(t\,y)\,dy = 0$$

is not exact, but that it becomes exact upon multiplying by

$$\mu(t, y) = \frac{1}{t\,y[f(t\,y) - g(t\,y)]}.$$

We see that

$$\begin{aligned}
\frac{\partial}{\partial y}\left(y f(t\,y)\right) &= f(t\,y) + ty f'(t\,y) \\
&\neq g(t\,y) + ty g'(t\,y) \\
&= \frac{\partial}{\partial t}\left(t g(t\,y)\right)
\end{aligned}$$

in general. Suppose we multiply the ODE by $\mu(t, y)$ and check for exactness.

$$\begin{aligned}
\frac{\partial}{\partial y}\left(\frac{y f(t\,y)}{t\,y[f(t\,y) - g(t\,y)]}\right) &= \frac{\partial}{\partial y}\left(\frac{f(t\,y)}{t[f(t\,y) - g(t\,y)]}\right) \\
&= \frac{t^2 f'(t\,y)[f(t\,y) - g(t\,y)] - t f(t\,y)[t f'(t\,y) - t g'(t\,y)}{t^2[f(t\,y) - g(t\,y)]^2} \\
&= \frac{f(t\,y) g'(t\,y) - f'(t\,y) g(t\,y)}{[f(t\,y) - g(t\,y)]^2} \\
&= \frac{y^2 g'(t\,y)[f(t\,y) - g(t\,y)] - y g(t\,y)[y f'(t\,y) - y g'(t\,y)}{y^2[f(t\,y) - g(t\,y)]^2} \\
&= \frac{\partial}{\partial t}\left(\frac{g(t\,y)}{y[f(t\,y) - g(t\,y)]}\right) \\
&= \frac{\partial}{\partial t}\left(\frac{t g(t\,y)}{t y[f(t\,y) - g(t\,y)]}\right)
\end{aligned}$$

4. Solve the following initial value problem.

$$\begin{aligned} \frac{dy}{dt} &= \frac{\sin y}{t\cos y - \sin^2 y} \\ y(0) &= \frac{\pi}{2} \end{aligned}$$

We may rearrange terms in the ODE to put it in the form below.

$$\sin y\, dt + (\sin^2 y - t\cos y)\, dy = 0$$

Since

$$\begin{aligned} \frac{\partial}{\partial y}(\sin y) &= \cos y \\ &\neq -\cos y \\ &= \frac{\partial}{\partial t}(\sin^2 y - t\cos y) \end{aligned}$$

the equation is not exact as given. It may become exact after multiplication by the appropriate integrating factor. If we try an integrating factor of the form $\mu(y)$ then the ODE may be written as

$$\mu(y)\sin y\, dt + \mu(y)(\sin^2 y - t\cos y)\, dy = 0.$$

If this equation is exact then

$$\begin{aligned} \frac{\partial}{\partial y}\left(\mu(y)\sin y\right) &= \mu'(y)\sin y + \mu(y)\cos y \\ -\mu(y)\cos y &= \frac{\partial}{\partial t}\Big(\mu(y)(\sin^2 y - t\cos y)\Big) \\ \mu'(y) + (2\cot y)\mu y &= 0 \\ \frac{d}{dy}\Big(\mu(y)\sin^2 y\Big) &= 0 \\ \mu(y)\sin^2 y &= C \\ \mu(y) &= \frac{1}{\sin^2 y} \end{aligned}$$

where we have chosen $C = 1$. Now the exact ODE has the form

$$\frac{1}{\sin y}\, dt + \left(1 - \frac{t\cos y}{\sin^2 y}\right)\, dy = 0.$$

If we assume that y is an implicit function of t, then

$$\begin{aligned} \psi(t,y) &= \int \frac{1}{\sin y}\, dt \\ &= \frac{t}{\sin y} + h(y) \end{aligned}$$

where h is an arbitrary function of y.

$$\begin{aligned}\frac{\partial}{\partial y}\psi(t,y) &= \frac{\partial}{\partial y}\left(\frac{t}{\sin y}+h(y)\right)\\ &= -\frac{t\cos y}{\sin^2 y}+h'(y)\\ &= 1-\frac{t\cos y}{\sin^2 y}\end{aligned}$$

which implies $h'(y)=1$ and $h(y)=y$. Thus the implicit form of the solution to the ODE is

$$\frac{t}{\sin y}+y=C.$$

If we use the initial condition we obtain the solution

$$\frac{t}{\sin y}+y=\frac{\pi}{2}.$$

5. Solve the following ordinary differential equation.

$$\frac{dy}{dt} = \frac{3y^2 \cot t + \sin t \cos t}{2y}$$

We may rearrange terms in the ODE to put it in the form below.

$$(3y^2 \cot t + \sin t \cos t)\, dt + (-2y)\, dy = 0$$

Since

$$\begin{aligned} \frac{\partial}{\partial y}(3y^2 \cot t + \sin t \cos t) &= 6y \cot t \\ &\neq 0 \\ &= \frac{\partial}{\partial t}(-2y) \end{aligned}$$

the equation is not exact as given. It may become exact after multiplication by the appropriate integrating factor. If we try an integrating factor of the form $\mu(t)$ then the ODE may be written as

$$\mu(t)(3y^2 \cot t + \sin t \cos t)\, dt + \mu(t)(-2y)\, dy = 0$$

If this equation is exact then

$$\begin{aligned} \frac{\partial}{\partial y}\left(\mu(t)(3y^2 \cot t + \sin t \cos t)\right) &= \mu(t)(6y \cot t) \\ -2y\mu'(t) &= \frac{\partial}{\partial t}\left(\mu(t)(-2y)\right) \\ \mu'(t) + (3 \cot t)\mu(t) &= 0 \\ \frac{d}{dt}\left(\mu(t) \sin^3 t\right) &= 0 \\ \mu(t) \sin^3 t &= C \\ \mu(t) &= \frac{1}{\sin^3 t} \end{aligned}$$

where we have chosen $C = 1$. Now the exact ODE has the form

$$\left(\frac{3y^2 \cos t}{\sin^4 t} + \frac{\cos t}{\sin^2 t}\right) dt + \left(\frac{-2y}{\sin^3 t}\right) dy = 0.$$

If we assume that y is an implicit function of t, then

$$\begin{aligned} \psi(t, y) &= \int \left(\frac{3y^2 \cos t}{\sin^4 t} + \frac{\cos t}{\sin^2 t}\right) dt \\ &= -\frac{y^2}{\sin^3 t} - \frac{1}{\sin t} + h(y) \end{aligned}$$

where h is an arbitrary function of y.

$$\begin{aligned}\frac{\partial}{\partial y}\psi(t,y) &= \frac{\partial}{\partial y}\left(-\frac{y^2}{\sin^3 t}-\frac{1}{\sin t}+h(y)\right)\\ &= -\frac{2y}{\sin^3 t}+h'(y)\\ &= -\frac{2y}{\sin^3 t}\end{aligned}$$

which implies $h'(y)=0$ and $h(y)$ is a constant. Thus the implicit form of the solution to the ODE is

$$\frac{y^2}{\sin^3 t}+\frac{1}{\sin t}=C.$$