Millersville University **Name** **Answer Key**
Department of Mathematics
MATH 365, *Ordinary Differential Equations*, Homework 05
October 1, 2008

Answer the following questions by solving the appropriate second order differential equations. Answers without justifying work will receive no credit. Partial credit will be given as appropriate, do not leave any problem blank. Each problem is worth 10 points. Your completed assignment is due at class time on Friday, October 3, 2008.

1. Solve each of the following ODEs and IVPs.

   (a) $y'' - 3y' + 2y = 0$

   The characteristic equation is

$$\begin{aligned} r^2 - 3r + 2 &= 0 \\ (r-1)(r-2) &= 0 \\ r_1 = 1 \quad \text{and} \quad r_2 &= 2 \end{aligned}$$

   therefore the general solution to the ODE is

$$y(t) = c_1 e^t + c_2 e^{2t}.$$

   (b) $y'' + 2y' + 5y = 0$

   The characteristic equation is

$$\begin{aligned} r^2 + 2r + 5 &= 0 \\ r &= \frac{-2 \pm \sqrt{4 - 4(1)(5)}}{2} \\ &= -1 \pm 2i \end{aligned}$$

   where $i = \sqrt{-1}$. Therefore the general solution to the ODE is

$$y(t) = e^{-t}(c_1 \cos 2t + c_2 \sin 2t).$$

(c) $y'' - 4y' + 4y = 0$, $y(0) = 2$, $y'(0) = 1$.
The characteristic equation is

$$\begin{aligned} r^2 - 4r + 4 &= 0 \\ (r-2)^2 &= 0 \\ r_1 = r_2 &= 2 \end{aligned}$$

therefore the general solution to the ODE is

$$y(t) = c_1 e^{2t} + c_2 t e^{2t}.$$

Making use of the initial conditions, we may solve for c_1 and c_2.

$$\begin{aligned} y(0) &= c_1 = 2 \\ y'(0) &= 4 + c_2 = 1 \quad \implies \quad c_2 = -3 \end{aligned}$$

Thus the solution to the IVP is

$$y(t) = 2e^{2t} - 3te^{2t}.$$

(d) $4y'' - 4y' + y = 0$, $y(1) = 0$, $y'(1) = 1$.
The characteristic equation is

$$\begin{aligned} 4r^2 - 4r + 1 &= 0 \\ (2r-1)^2 &= 0 \\ r_1 = r_2 &= \frac{1}{2} \end{aligned}$$

therefore the general solution to the ODE is

$$y(t) = c_1 e^{t/2} + c_2 t e^{t/2}.$$

Making use of the initial conditions, we may solve for c_1 and c_2.

$$\begin{aligned} y(1) &= (c_1 + c_2)e^{1/2} = 0 \quad \implies \quad c_2 = -c_1 \\ y'(1) &= \frac{1}{2}(c_1 + 3c_2)e^{1/2} = 1 \\ &= (c_1 - 3c_1)e^{1/2} = 2 \\ &= -c_1 e^{1/2} = 1 \end{aligned}$$

Hence $c_1 = -e^{-1/2}$ and $c_2 = e^{-1/2}$.
Thus the solution to the IVP is

$$y(t) = -e^{-1/2}e^{t/2} + e^{-1/2}te^{t/2} = te^{(t-1)/2} - e^{(t-1)/2}.$$

2. Show that $y_1(t) = \cos t$ is a solution to the following ODE

$$y'' + (\cos t)y' + (1 + \sin t)y = 0$$

and find a second, linearly independent solution.

$$\begin{aligned}
(\cos t)'' + (\cos t)(cost)' + (1 + \sin t)\cos t &= -\cos t - \cos t \sin t + \cos t + \cos t \sin t \\
&= 0
\end{aligned}$$

Thus $y_1(t) = \cos t$ is a solution to the ODE.

We can use reduction of order to find the second solution.

$$\begin{aligned}
y_2(t) &= v(t)\cos t \\
y_2'(t) &= v'(t)\cos t - v(t)\sin t \\
y_2''(t) &= v''(t)\cos t - 2v'(t)\sin t - v(t)\cos t
\end{aligned}$$

Substituting into the ODE yields

$$\begin{aligned}
0 &= v''\cos t - 2v'\sin t - v\cos t + (\cos t)(v'\cos t - v\sin t) + (1 + \sin t)v\cos t \\
&= v''\cos t + (-2\sin t + \cos^2 t)v' + (-\cos t - \cos t\sin t + \cos t + \cos t\sin t)v \\
&= v''\cos t + (\cos^2 t - 2\sin t)v' \\
&= v'' + (\cos t - 2\tan t)v' \\
0 &= w' + (\cos t - 2\tan t)w
\end{aligned}$$

where $w = v'$. Solving this first order linear ODE for w involves the integrating factor

$$\mu(t) = e^{\int(\cos t - 2\tan t)\,dt} = e^{\sin t + 2\ln|\cos t|} = \cos^2 t\, e^{\sin t}.$$

Now we have

$$\begin{aligned}
w' + (\cos t - 2\tan t)w &= 0 \\
\frac{d}{dt}(\mu(t)w) &= 0 \\
\mu(t)w(t) &= C \\
w(t) &= \sec^2 t\, e^{-\sin t}
\end{aligned}$$

where we have chosen $C = 1$. Then

$$v(t) = \int \sec^2 t\, e^{-\sin t}\,dt$$

and

$$y_2(t) = \cos t \int \sec^2 t\, e^{-\sin t}\,dt.$$

3. Consider the IVP

$$\begin{aligned} y'' + 2ay' + (a^2+1)y &= 0 \\ y(0) &= 1 \\ y'(0) &= -1 \end{aligned}$$

If $a = \pi$ find the smallest value of T such that $|y(t)| < 1/100$ for all $t > T$.

The characteristic equation is

$$\begin{aligned} r^2 + 2\pi r + (\pi^2 + 1) &= 0 \\ r &= \frac{-2\pi \pm \sqrt{4\pi^2 - 4(1)(\pi^2+1)}}{2} \\ &= -\pi \pm i \end{aligned}$$

where $i = \sqrt{-1}$. The general solution to the ODE is

$$y(t) = e^{-\pi t}(c_1 \cos t + c_2 \sin t).$$

Making use of the initial conditions, we may solve for c_1 and c_2.

$$\begin{aligned} y(0) &= c_1 = 1 \\ y'(0) &= -\pi + c_2 = -1 \quad \Longrightarrow \quad c_2 = \pi - 1 \end{aligned}$$

Thus the solution to the IVP is

$$y(t) = e^{-\pi t}(\cos t + (\pi - 1)\sin t).$$

Using Newton's Method with an initial estimate of $T \approx 1.5$ we find that $|y(t)| < 1/100$ for all $t > 1.68807$.

4. Find the general form of the Wronskian for the ODE

$$t^2 y'' + t\cos(\ln t)y' + (t^2 - n^2)y = 0$$

where $n \in \mathbb{N}$.

After re-writing the ODE in the form

$$y'' + \frac{1}{t}\cos(\ln t)y' + \left(1 - \frac{n^2}{t^2}\right)y = 0$$

we may use Abel's Theorem to find the general form of the Wronskian.

$$\begin{aligned} W &= Ce^{-\int p(t)\,dt} \\ &= Ce^{-\int \frac{1}{t}\cos(\ln t)\,dt} \\ &= Ce^{-\sin(\ln t)} \end{aligned}$$