Millersville University **Name** **Answer Key**
Department of Mathematics
MATH 365, *Ordinary Differential Equations*, Homework 06
October 13, 2008

Find the solutions to the following ODEs and IVPs. Answers without justifying work will receive no credit. Partial credit will be given as appropriate, do not leave any problem blank. Each problem is worth 10 points. Your completed assignment is due at class time on Wednesday, October 15, 2008.

1. $y'' + 9y = \csc 3t$

   By use of the characteristic equation and the quadratic formula we may find the complementary solution $y_c(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 \cos 3t + c_2 \sin 3t$. Using the variation of parameters formula then

$$\begin{aligned}
\mu_1'(t) &= -\frac{g(t)y_2(t)}{W(y_1,y_2)(t)} \\
&= -\frac{\csc 3t \sin 3t}{3} \\
&= -\frac{1}{3} \\
\mu_1(t) &= -\frac{1}{3}t,
\end{aligned}$$

   and

$$\begin{aligned}
\mu_2'(t) &= \frac{g(t)y_1(t)}{W(y_1,y_2)(t)} \\
&= \frac{\csc 3t \cos 3t}{3} \\
&= \frac{1}{3}\cot 3t \\
\mu_2(t) &= \frac{1}{9}\ln|\sin 3t|.
\end{aligned}$$

   Thus the general solution to the ODE is

$$y(t) = c_1 \cos 3t + c_2 \sin 3t - \frac{1}{3}t\cos 3t + \frac{1}{9}\ln|\sin 3t|\sin 3t.$$

2. $y'' - y = e^{-t^2}$, $y(0) = 1$, $y'(0) = -1$

By use of the characteristic equation and the quadratic formula we may find the complementary solution $y_c(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 e^t + c_2 e^{-t}$. Using the variation of parameters formula then

$$\begin{aligned} \mu_1'(t) &= -\frac{g(t)y_2(t)}{W(y_1,y_2)(t)} \\ &= -\frac{e^{-t^2}e^{-t}}{-2} \\ &= \frac{1}{2}e^{-t-t^2} \\ \mu_1(t) &= \frac{1}{2}\int_0^t e^{-s-s^2}\,ds, \end{aligned}$$

and

$$\begin{aligned} \mu_2'(t) &= \frac{g(t)y_1(t)}{W(y_1,y_2)(t)} \\ &= \frac{e^{-t^2}e^{t}}{-2} \\ &= -\frac{1}{2}e^{t-t^2} \\ \mu_2(t) &= -\frac{1}{2}\int_0^t e^{s-s^2}\,ds. \end{aligned}$$

Therefore the general solution to the ODE has the form:

$$y(t) = c_1 e^t + c_2 e^{-t} + \frac{e^t}{2}\int_0^t e^{-s-s^2}\,ds - \frac{e^{-t}}{2}\int_0^t e^{s-s^2}\,ds$$

Using the initial conditions we see that

$$\begin{aligned} y(0) &= 1 = c_1 + c_2 \\ y'(0) &= -1 = c_1 - c_2 \end{aligned}$$

which implies that $c_1 = 0$ and $c_2 = 1$. Finally we have the solution

$$y(t) = e^{-t} + \frac{e^t}{2}\int_0^t e^{s-s^2}\,ds - \frac{e^{-t}}{2}\int_0^t e^{-s-s^2}\,ds$$

3. $$y'' + 3y' + 2y = 3e^{-2t} + t$$

By use of the characteristic equation and the quadratic formula we may find the complementary solution $y_c(t) = c_1y_1(t) + c_2y_2(t) = c_1e^{-t} + c_2e^{-2t}$. We will use the method of undetermined coefficients to find a particular solution. Suppose

$$\begin{aligned} Y(t) &= Ate^{-2t} + Bt + C \\ Y'(t) &= A(1-2t)e^{-2t} + B \\ Y''(t) &= 4A(t-1)e^{-2t}, \end{aligned}$$

then

$$\begin{aligned} 3e^{-2t} + t &= Y'' + 3Y' + 2Y \\ &= 4A(t-1)e^{-2t} + 3A(1-2t)e^{-2t} + 3B \\ &\quad + 2Ate^{-2t} + 2Bt + 2C \\ &= -Ae^{-2t} + 2Bt + 3B + 2C. \end{aligned}$$

Thus $A = -3$, $B = 1/2$, and $C = -3/4$. Consequently the general solution is

$$y(t) = c_1e^{-t} + c_2e^{-2t} - 3te^{-2t} + \frac{t}{2} - \frac{3}{4}.$$

4. $\quad y'' + y' + 9y = t^2 e^t$

The roots of the characteristic equation are

$$r = -\frac{1}{2} \pm i\frac{\sqrt{35}}{2}$$

and thus the complementary solution is

$$y_c(t) = e^{-t/2}\left(c_1 \cos\frac{\sqrt{35}}{2}t + c_2 \sin\frac{\sqrt{35}}{2}t\right).$$

We will use the method of undetermined coefficients to find the particular solution. If we assume

$$\begin{aligned} Y(t) &= (At^2 + bt + C)e^t \\ Y'(t) &= (At^2 + (2A + B)t + B + C)e^t \\ Y''(t) &= (At^2 + (4A + B)t + 2A + 2B + C)e^t \end{aligned}$$

then

$$\begin{aligned} t^2 e^t &= (At^2 + (4A + B)t + 2A + 2B + C)e^t \\ &\quad + (At^2 + (2A + B)t + B + C)e^t + 9(At^2 + bt + C)e^t \\ &= (11At^2 + (6A + 11B)t + 2A + 3B + 11C)e^t \\ t^2 &= 11At^2 + (6A + 11B)t + 2A + 3B + 11C \end{aligned}$$

which implies $A = 1/11$. Substituting this value into the equation above yields

$$\begin{aligned} t^2 &= t^2 + \left(\frac{6}{11} + 11B\right)t + \frac{2}{11} + 3B + 11C \\ 0 &= \left(\frac{6}{11} + 11B\right)t + \frac{2}{11} + 3B + 11C \end{aligned}$$

which implies $B = -6/121$. Substituting this value into the equation above yields

$$\begin{aligned} 0 &= \left(\frac{6}{11} + 11\left[-\frac{6}{121}\right]\right)t + \frac{2}{11} - \frac{18}{121} + 11C \\ 0 &= \frac{2}{11} - \frac{18}{121} + 11C = \frac{22}{121} - \frac{18}{121} + 11C \\ -\frac{4}{121} &= 11C \\ C &= -\frac{4}{1331}. \end{aligned}$$

Thus the general solution to the ODE is

$$y(t) = e^{-t/2}\left(c_1 \cos\frac{\sqrt{35}}{2}t + c_2 \sin\frac{\sqrt{35}}{2}t\right) + \left(\frac{1}{11}t^2 - \frac{6}{121}t - \frac{4}{1331}\right)e^t$$

5. $y'' + y' - 2y = \ln t$

Using the characteristic equation we can find the complementary solution to the ODE is $y_c(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 e^{-2t} + c_2 e^{t}$. Using the variation of parameters formula then

$$\begin{aligned}
\mu_1'(t) &= -\frac{g(t)y_2(t)}{W(y_1,y_2)(t)} \\
&= -\frac{(\ln t)e^{t}}{3e^{-t}} \\
&= -\frac{1}{3}(\ln t)e^{2t} \\
\mu_1(t) &= -\frac{1}{3}\int_1^t (\ln s)e^{2s}\,ds,
\end{aligned}$$

and

$$\begin{aligned}
\mu_2'(t) &= \frac{g(t)y_1(t)}{W(y_1,y_2)(t)} \\
&= \frac{(\ln t)e^{-2t}}{3e^{-t}} \\
&= \frac{1}{3}(\ln t)e^{-t} \\
\mu_2(t) &= \frac{1}{3}\int_1^t (\ln s)e^{-s}\,ds.
\end{aligned}$$

Thus the general solution to the ODE is

$$y(t) = c_1 e^{-2t} + c_2 e^{t} - \frac{1}{3}e^{-2t}\int_1^t (\ln s)e^{2s}\,ds + \frac{1}{3}e^{t}\int_1^t (\ln s)e^{-s}\,ds.$$

6. Show that $y = c_1 t^2 + c_2 t$ is a complementary solution of

$$t^2 y'' - 2ty' + 2y = te^{-t}.$$

Find the general solution to this nonhomogeneous ODE.

Let $y_1(t) = t^2$ and $y_2(t) = t$, then

$$\begin{aligned} t^2(t^2)'' - 2t(t^2)' + 2(t^2) &= t^2(2) - 2t(2t) + 2t^2 \\ &= 2t^2 - 4t^2 + 2t^2 \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} t^2(t)'' - 2t(t)' + 2(t) &= t^2(0) - 2t(1) + 2t \\ &= -2t + 2t \\ &= 0, \end{aligned}$$

which shows that $y_1(t)$ and $y_2(t)$ are solutions to the corresponding homogeneous equation. The Wronskian is $W(y_1, y_2)(t) = -t^2$ which demonstrates the solutions are linearly independent.

We can re-write the nonhomogeneous ODE in the form

$$y'' - \frac{2}{t}y' + \frac{2}{t^2}y = \frac{1}{t}e^{-t}.$$

Using the variation of parameters formula we see that

$$\begin{aligned} \mu_1'(t) &= -\frac{g(t)y_2(t)}{W(y_1, y_2)(t)} = -\frac{(\frac{1}{t})te^{-t}}{-t^2} = \frac{e^{-t}}{t^2} \\ \mu_1(t) &= \int_1^t \frac{e^{-s}}{s^2}\,ds, \end{aligned}$$

and

$$\begin{aligned} \mu_2'(t) &= \frac{g(t)y_1(t)}{W(y_1, y_2)(t)} = \frac{(\frac{1}{t})t^2e^{-t}}{-t^2} = -\frac{e^{-t}}{t} \\ \mu_2(t) &= -\int_1^t \frac{e^{-s}}{s}\,ds. \end{aligned}$$

Thus the general solution to the ODE is

$$y(t) = c_1 t^2 + c_2 t + t^2 \int_1^t \frac{e^{-s}}{s^2}\,ds - t \int_1^t \frac{e^{-s}}{s}\,ds.$$