Millersville University **Name** **Answer Key**
Department of Mathematics
MATH 365, *Ordinary Differential Equations*, Homework 07
October 20, 2008

Find the solutions to the following exercises. Answers without justifying work will receive no credit. Partial credit will be given as appropriate, do not leave any problem blank. Each problem is worth 10 points. Your completed assignment is due at class time on Friday, October 24, 2008.

1. When a mass m is at the end of a vertically hanging spring, the period is 1.5 seconds. After adding 3 kilograms to the mass, the period becomes 2.5 seconds. What was the original mass m?

$$\begin{aligned} T &= \frac{3}{2} = \frac{2\pi}{\sqrt{\frac{k}{m}}} \\ \frac{2}{3} &= \frac{\sqrt{\frac{k}{m}}}{2\pi} \\ \frac{4}{9} &= \frac{\frac{k}{m}}{4\pi^2} \\ k &= \frac{16\pi^2 m}{9} \end{aligned}$$

After increasing the mass

$$\begin{aligned} \frac{5}{2} &= \frac{2\pi}{\sqrt{\frac{k}{m+3}}} \\ \frac{2}{5} &= \frac{\sqrt{\frac{k}{m+3}}}{2\pi} \\ \frac{4}{25} &= \frac{\frac{k}{m+3}}{4\pi^2} \\ \frac{16\pi^2}{25} &= \frac{k}{m+3} \\ m+3 &= \frac{25k}{16\pi^2} \\ &= \frac{25m}{9} \\ m &= \frac{27}{16} \quad \text{kg.} \end{aligned}$$

2. If a hole were bored through the earth's center, one would find that an object dropped in the hole is acted upon by a force of attraction varying directly as the distance between the object and the earth's center. Assuming the earth is a sphere with a radius of 4000 miles, find the period of motion of the object dropped in the hole and the velocity of the object as it passes through the earth's center.

   For the sake of convenience we will convert all distances to miles. At the earth's surface the attraction due to gravity is

$$g_{4000} = \frac{32}{5280} \quad \text{mi/s}^2$$

   and at the center of the earth $g_0 = 0$. Thus the position variable gravitational attraction can be expressed as

$$g_u = \frac{32u}{(5280)(4000)} = \frac{u}{660000}.$$

   Hence the motion of the object dropped in the hole is governed by the IVP:

$$\begin{aligned} mu'' + \frac{m}{660000}u &= 0 \\ u(0) &= -4000 \\ u'(0) &= 0 \end{aligned}$$

   The period is

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{\sqrt{\frac{1}{660000}}} = 200\sqrt{66}\pi \approx 5104.48 \text{ s} \approx 1.41791 \text{ hr.}$$

   The position is given by the solution to the IVP:

$$u(t) = -4000\cos\left(\frac{t}{100\sqrt{66}}\right).$$

   The object is at the center of the earth when

$$\frac{t}{100\sqrt{66}} = \frac{\pi}{2} \quad \Longrightarrow \quad t = 50\sqrt{66}\pi.$$

   Consequently

$$u'(50\sqrt{66}\pi) = 20\sqrt{\frac{2}{33}} \approx 4.92366 \text{ mi/s} \approx 25996.9 \text{ ft/s} \approx 17725.2 \text{ mph.}$$

3. Suppose that the position of a simple harmonic oscillator is described by the function $u(t) = Re^{-\alpha t}\cos(\omega t - \delta)$. Find the times at which the position is a local extremum.

We can find the critical numbers of $u(t)$ by differentiating $u(t)$ and setting the derivative equal to 0.

$$\begin{aligned}
u'(t) &= -Re^{-\alpha t}\left(\alpha\cos(\omega t - \delta) + \omega\sin(\omega t - \delta)\right) \\
0 &= \alpha\cos(\omega t - \delta) + \omega\sin(\omega t - \delta) \\
-\alpha\cos(\omega t - \delta) &= \omega\sin(\omega t - \delta) \\
-\frac{\alpha}{\omega} &= \tan(\omega t - \delta) \\
-\tan^{-1}\left(\frac{\alpha}{\omega}\right) + n\pi &= \omega t_n - \delta \\
\delta - \tan^{-1}\left(\frac{\alpha}{\omega}\right) + n\pi &= \omega t_n \\
t_n &= \frac{1}{\omega}\left[\delta - \tan^{-1}\left(\frac{\alpha}{\omega}\right) + n\pi\right]
\end{aligned}$$

for $n \in \mathbb{Z}$.

4. Suppose that the position of a simple harmonic oscillator is described by the function $u(t) = Re^{-\alpha t}\cos(\omega t - \delta)$. Suppose $u(t_n)$ and $u(t_{n+1})$ are two successive local extreme displacements from equilibrium. Find the ratio $\dfrac{u(t_{n+1})}{u(t_n)}$.

Consider $u(t_k)$.

$$\begin{aligned} u(t_k) &= Re^{-\alpha t_k}\cos(\omega t_k - \delta) \\ &= Re^{-\alpha t_k}\cos\left(\tan^{-1}\left(\frac{\alpha}{\omega}\right) + k\pi\right) \\ &= (-1)^k Re^{-\alpha t_k}\cos\left(\tan^{-1}\left(\frac{\alpha}{\omega}\right)\right) \\ &= (-1)^k Re^{-\alpha t_k}\frac{\omega}{\sqrt{\omega^2+\alpha^2}} \end{aligned}$$

Therefore

$$\begin{aligned} \frac{u(t_{n+1})}{u(t_n)} &= \frac{(-1)^{n+1}Re^{-\alpha t_{n+1}}\frac{\omega}{\sqrt{\omega^2+\alpha^2}}}{(-1)^n Re^{-\alpha t_n}\frac{\omega}{\sqrt{\omega^2+\alpha^2}}} \\ &= -\frac{e^{-\alpha t_{n+1}}}{e^{-\alpha t_n}} \\ &= -e^{-\alpha(t_{n+1}-t_n)}. \end{aligned}$$

We can see that

$$\begin{aligned} -\alpha(t_{n+1} - t_n) &= -\alpha\left(\frac{1}{\omega}\left[\delta - \tan^{-1}\left(\frac{\alpha}{\omega}\right) + (n+1)\pi\right] - \frac{1}{\omega}\left[\delta - \tan^{-1}\left(\frac{\alpha}{\omega}\right) + n\pi\right]\right) \\ &= -\frac{\alpha\pi}{\omega}. \end{aligned}$$

Consequently

$$\frac{u(t_{n+1})}{u(t_n)} = -e^{-\frac{\alpha\pi}{\omega}}.$$

5. Suppose the motion of a forced, undamped simple harmonic oscillator is described by the IVP

$$\begin{aligned} mu'' + ku &= \begin{cases} t & \text{if } 0 \le t \le T \\ 0 & \text{if } t \ge T \end{cases} \\ u(0) &= 0 \\ u'(0) &= 0. \end{aligned}$$

Find the solution to the IVP.

The complementary solution to the ODE is $u(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$ where $\omega_0 = \sqrt{k/m}$.

Since the nonhomogeneous forcing is piecewise defined we can give a piecewise defined solution to the IVP. For $t \in [0, T]$ we may use the method of undetermined coefficients to find the particular solution. Assume $U(t) = At + B$, differentiate this function and substitute into the ODE when $0 \le t \le T$.

$$\begin{aligned} mU'' + kU &= t \\ m(0) + k(At + B) &= t \\ kAt + kB &= t \end{aligned}$$

We can see that $A = 1/k$ and $B = 0$. Thus the general solution for $t \in [0, T]$ has the form

$$u(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{t}{k}.$$

Making use of the initial conditions allows us to see that $u(0) = 0 = c_1$ and

$$u'(0) = 0 = \frac{1}{k} + c_2 \omega_0$$

which implies that $c_2 = -1/(k\omega_0)$. Therefore on the interval $[0, T]$ the solution to the IVP is

$$u(t) = \frac{t}{k} - \frac{1}{k\omega_0} \sin \omega_0 t.$$

For $t \ge T$ we can use the complementary solution (since the nonhomogeneous forcing is 0) and the new initial conditions

$$\begin{aligned} u(T) &= \frac{T}{k} - \frac{1}{k\omega_0} \sin \omega_0 T \\ u'(T) &= \frac{1}{k} - \frac{1}{k} \cos \omega_0 T. \end{aligned}$$

Solving the system of equations

$$\begin{aligned} c_1 \cos \omega_0 T + c_2 \sin \omega_0 T &= \frac{T}{k} - \frac{1}{k\omega_0} \sin \omega_0 T \\ -c_1 \omega_0 \sin \omega_0 T + c_2 \omega_0 \cos \omega_0 T &= \frac{1}{k} \cos \omega_0 T \end{aligned}$$

we find that

$$\begin{aligned} c_1 &= \frac{\omega_0 T \cos \omega_0 T - \sin \omega_0 T}{k\omega_0} \\ c_2 &= \frac{\omega_0 T \sin \omega_0 T + \cos \omega_0 T - 1}{k\omega_0}. \end{aligned}$$

Thus we may summarize the piecewise defined solution to the IVP as

$$u(t) = \begin{cases} \frac{t}{k} - \frac{1}{k\omega_0} \sin \omega_0 t & \text{if } t \in [0, T], \\ \left(\frac{\omega_0 T \cos \omega_0 T - \sin \omega_0 T}{k\omega_0}\right) \cos \omega_0 t + \left(\frac{\omega_0 T \sin \omega_0 T + \cos \omega_0 T - 1}{k\omega_0}\right) \sin \omega_0 t & \text{if } t \geq T. \end{cases}$$

Using the difference of angles formula for sine and cosine we may simplify this to Thus we may summarize the piecewise defined solution to the IVP as

$$u(t) = \begin{cases} \frac{t}{k} - \frac{1}{k\omega_0} \sin \omega_0 t & \text{if } t \in [0, T], \\ \frac{T}{k} \cos \omega_0 (t - T) + \frac{1}{k\omega_0} \sin \omega_0 (t - T) - \frac{1}{k\omega_0} \sin \omega_0 t & \text{if } t \geq T. \end{cases}$$

If we wish we may re-write the solution using the product-to-sum formula.

$$u(t) = \begin{cases} \frac{t}{k} - \frac{1}{k\omega_0} \sin \omega_0 t & \text{if } t \in [0, T], \\ \frac{T}{k} \cos \omega_0 (t - T) - \frac{2}{k\omega_0} \sin \omega_0 \frac{T}{2} \cos \omega_0 \left(t - \frac{T}{2}\right) & \text{if } t \geq T. \end{cases}$$

6. A mass on a spring undergoes a forced vibration given by

$$mu'' + ku = F_0 \cos^3 \omega t.$$

Show that there are two values of ω at which resonance occurs and find them.

We can use the product-to-sum formula for cosine to write

$$\begin{aligned}
\cos^3 \omega t &= (\cos \omega t \cos \omega t) \cos \omega t \\
&= \frac{1}{2}(1 + \cos 2\omega t) \cos \omega t \\
&= \frac{1}{2} \cos \omega t + \frac{1}{2} \cos 2\omega t \cos \omega t \\
&= \frac{1}{2} \cos \omega t + \frac{1}{4}(\cos \omega t + \cos 3\omega t) \\
&= \frac{3}{4} \cos \omega t + \frac{1}{4} \cos 3\omega t.
\end{aligned}$$

Thus if $\omega = \omega_0 = \sqrt{k/m}$ resonance will occur. Also if

$$\begin{aligned}
3\omega &= \sqrt{k/m} \\
\omega &= \frac{1}{3}\sqrt{k/m}
\end{aligned}$$

resonance will occur.