Millersville University **Name** **Answer Key**

Department of Mathematics

MATH 365, *Ordinary Differential Equations*, Homework 08

November 5, 2008

Find the solutions to the following exercises. Answers without justifying work will receive no credit. Partial credit will be given as appropriate, do not leave any problem blank. Each problem is worth 10 points. Your completed assignment is due at class time on Friday, November 7, 2008.

1. Solve each of the following ODEs/IVPs.

   (a) $t^2y'' - 2ty' + 2y = 0$

   Making the change of variable $t = e^z$ yields the second order linear constant coefficient ODE:

   $$\frac{d^2y}{dz^2} - 3\frac{dy}{dz} + 2y = 0$$

   with characteristic equation

   $$0 = r^2 - 3r + 2 = (r-1)(r-2).$$

   Thus $r_1 = 1$ and $r_2 = 2$. The general solution to the Euler equation is

   $$y(t) = c_1 t + c_2 t^2.$$

   (b) $4t^2y'' + y = 0$; $y(1) = 1$, $y'(1) = 1$

   Re-writing the Euler equation as

   $$t^2y'' + (0)ty' + \frac{1}{4}y = 0$$

   and making the change of variable $t = e^z$ yields the second order linear constant coefficient ODE:

   $$\frac{d^2y}{dz^2} - \frac{dy}{dz} + \frac{1}{4}y = 0$$

   with characteristic equation

   $$0 = r^2 - r + \frac{1}{4} = \left(r - \frac{1}{2}\right)^2.$$

   Thus $r_1 = r_2 = r = 1/2$. The general solution to the Euler equation is

   $$y(t) = (c_1 + c_2 \ln t)\, t^{1/2}.$$

Making use of the initial conditions we have

$$\begin{aligned} y(1) &= 1 = c_1 \\ y'(1) &= 1 = c_2 + \frac{1}{2} \end{aligned}$$

which implies $c_2 = 1/2$. Therefore the solution to the IVP is

$$y(t) = \left(1 + \frac{1}{2}\ln t\right) t^{1/2}.$$

(c) $t^2 y'' - 2y = t$

Making the change of variable $t = e^z$ yields the second order linear constant coefficient ODE:

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} - 2y = 0$$

with characteristic equation

$$0 = r^2 - r - 2 = (r-2)(r+1)$$

Thus $r_1 = -1$ and $r_2 = 2$. The complementary solution to the Euler equation is

$$y_c(t) = \frac{c_1}{t} + c_2 t^2.$$

If we apply the same change of variable to the right-hand side of the nonhomogeneous Euler equation we obtain the ODE

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} - 2y = e^z$$

which can be solved using the method of undetermined coefficients. If we assume $Y(z) = Ae^z$ then

$$\begin{aligned} Ae^z - Ae^z - 2Ae^z &= e^z \\ -2A &= 1 \\ A &= -\frac{1}{2}. \end{aligned}$$

Therefore the general solution to the Euler equation is

$$y(t) = \frac{c_1}{t} + c_2 t^2 - \frac{t}{2}.$$

(d) $t^2y'' + y = 16\sin(\ln t)$

Making the change of variable $t = e^z$ yields the second order linear constant coefficient ODE:

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} + y = 0$$

with characteristic equation

$$0 = r^2 - r + 1.$$

Thus $r_{1,2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ where $i = \sqrt{-1}$. The complementary solution to the Euler equation is

$$y_c(t) = t^{1/2}\left(c_1\cos\left(\frac{\sqrt{3}}{2}\ln t\right) + c_2\sin\left(\frac{\sqrt{3}}{2}\ln t\right)\right).$$

If we apply the same change of variable to the right-hand side of the nonhomogeneous Euler equation we obtain the ODE

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} + y = 16\sin z$$

which can be solved using the method of undetermined coefficients. If we assume $Y(z) = A\cos z + B\sin z$ then

$$\begin{aligned} -A\cos z - B\sin z - (-A\sin z + B\cos z) + A\cos z + B\sin z &= 16\sin z \\ (-A - B + A)\cos z + (-B + A + B)\sin z &= \\ -B\cos z + A\sin z &= \end{aligned}$$

which implies $B = 0$ and $A = 16$. Therefore the general solution to the Euler equation is

$$y(t) = t^{1/2}\left(c_1\cos\left(\frac{\sqrt{3}}{2}\ln t\right) + c_2\sin\left(\frac{\sqrt{3}}{2}\ln t\right)\right) + 16\cos(\ln t).$$

2. Consider the ODE

$$(2t+3)^2 y'' + (2t+3)y' - 2y = 0.$$

Making the change of variable $2t+3 = e^z$, solve the ODE above.

If we let $2t+3 = e^z$ then $t = (e^z - 3)/2$ and using the chain rule for derivatives

$$\frac{dy}{dz} = \frac{dy}{dt}\frac{dt}{dz} = \frac{1}{2}y'e^z$$

which implies

$$y' = \frac{2}{e^z}\frac{dy}{dz} = \frac{2}{2t+3}\frac{dy}{dz}.$$

Taking the second derivative we obtain

$$\frac{d^2y}{dz^2} = \frac{d}{dz}\left(\frac{dy}{dz}\right) = \frac{d}{dz}\left(\frac{1}{2}y'e^z\right) = \frac{1}{4}y''e^{2z} + \frac{1}{2}y'e^z.$$

This implies

$$y'' = \frac{4}{(2t+3)^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right).$$

Substituting into the ODE produces

$$\begin{aligned}
0 &= (2t+3)^2\frac{4}{(2t+3)^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) + (2t+3)\frac{2}{2t+3}\frac{dy}{dz} - 2y \\
&= 4\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) + 2\frac{dy}{dz} - 2y \\
&= 4\frac{d^2y}{dz^2} - 2\frac{dy}{dz} - 2y \\
&= 2\frac{d^2y}{dz^2} - \frac{dy}{dz} - y.
\end{aligned}$$

This second order linear constant coefficient homogeneous ODE has characteristic equation

$$0 = 2r^2 - r - 1 = (2r+1)(r-1)$$

and thus $r_1 = -1/2$ and $r_2 = 1$. Consequently the solution is

$$\begin{aligned}
y(z) &= c_1e^{-z/2} + c_2e^z \\
y(t) &= c_1(2t+3)^{-1/2} + c_2(2t+3).
\end{aligned}$$

3. Consider the ODE

$$y'' + (\tan t)y' + (\cos^2 t)y = 0.$$

Making the change of variable $z = \sin t$, solve the ODE above.

If we let $z = \sin t$ and use the chain rule for derivatives we find

$$\begin{aligned}
\frac{dy}{dt} &= \frac{dy}{dz}\frac{dz}{dt} = \frac{dy}{dz}\cos t \\
\frac{d^2y}{dt^2} &= \frac{d}{dt}\left(\frac{dy}{dz}\cos t\right) = \frac{d^2y}{dz^2}\cos^2 t - \frac{dy}{dz}\sin t.
\end{aligned}$$

Substituting into the ODE yields

$$\begin{aligned}
0 &= \frac{d^2y}{dz^2}\cos^2 t - \frac{dy}{dz}\sin t + (\tan t)\frac{dy}{dz}\cos t + (\cos^2 t)y \\
&= \frac{d^2y}{dz^2}\cos^2 t - \frac{dy}{dz}\sin t + \frac{dy}{dz}\sin t + (\cos^2 t)y \\
&= \frac{d^2y}{dz^2}\cos^2 t + (\cos^2 t)y \\
&= \frac{d^2y}{dz^2} + y.
\end{aligned}$$

This second order linear constant coefficient homogeneous ODE has characteristic equation

$$0 = r^2 + 1$$

and thus $r_{1,2} = \pm i$ where $i = \sqrt{-1}$. Consequently the solution is

$$\begin{aligned}
y(z) &= c_1 \cos z + c_2 \sin z \\
y(t) &= c_1 \cos(\sin t) + c_2 \sin(\sin t).
\end{aligned}$$