Millersville University **Name** **Answer Key**
Department of Mathematics
MATH 365, *Ordinary Differential Equations*, Homework 09
November 12, 2008

Find the solutions to the following exercises. Answers without justifying work will receive no credit. Partial credit will be given as appropriate, do not leave any problem blank. Each problem is worth 10 points. Your completed assignment is due at class time on Friday, November 14, 2008.

1. Show that $\frac{d}{dx}[xJ_1(x)] = xJ_0(x)$.

$$\begin{aligned}
\frac{d}{dx}[xJ_1(x)] &= \frac{d}{dx}\left[\frac{x^2}{2}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{4^n(n+1)!(n!)}\right] \\
&= \frac{d}{dx}\left[\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{2^{2n+1}(n+1)!(n!)}\right] \\
&= \sum_{n=0}^{\infty}\frac{d}{dx}\left[\frac{(-1)^n x^{2n+2}}{2^{2n+1}(n+1)!(n!)}\right] \\
&= \sum_{n=0}^{\infty}\frac{(2n+2)(-1)^n x^{2n+1}}{2^{2n+1}(n+1)!(n!)} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2^{2n}(n)!(n!)} \\
&= x\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{4^n(n!)^2} \\
&= x\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{4^n(n!)^2}\right] \\
&= xJ_0(x)
\end{aligned}$$

2. Evaluate the indefinite integral

$$\int x\, J_0(x)\, dx.$$

By the previous exercise $\dfrac{d}{dx}[xJ_1(x)] = xJ_0(x)$, thus $xJ_1(x)$ is an antiderivative of $xJ_0(x)$. Therefore

$$\int x\, J_0(x)\, dx = xJ_1(x) + C.$$

3. Evaluate the indefinite integral

$$\int x^3 \, J_0(x) \, dx.$$

$$\int x^3 \, J_0(x) \, dx = \int x^2 \left(x J_0(x) \right) \, dx$$

Using integration by parts with

$$\begin{aligned} u &= x^2 & \qquad v &= x J_1(x) \\ du &= 2x \, dx & \qquad dv &= x J_0(x) \, dx \end{aligned}$$

we obtain

$$\begin{aligned} \int x^3 \, J_0(x) \, dx &= \int x^2 \left(x J_0(x) \right) \, dx \\ &= x^3 J_1(x) - \int 2x^2 J_1(x) \, dx \\ &= x^3 J_1(x) - 2 \int x^2 J_1(x) \, dx \\ &= x^3 J_1(x) - 2x^2 J_2(x) + C \\ &= x^3 J_1(x) - 2x^2 \left(\frac{2}{x} J_1(x) - J_0(x) \right) + C \\ &= x^3 J_1(x) - 4x J_1(x) + 2x^2 J_0(x) + C \\ &= (x^3 - 4x) J_1(x) + 2x^2 J_0(x) + C \end{aligned}$$

4. Find the general solution of the ODE:

$$x^2y'' + xy' + (x^2 - 2)y = 0$$

The value $x = 0$ is a regular singular point since

$$\begin{aligned}
\lim_{x\to 0} x\left(\frac{x}{x^2}\right) &= 1 = p_0 \\
\lim_{x\to 0} x^2\left(\frac{x^2-2}{x^2}\right) &= -2 = q_0.
\end{aligned}$$

The exponents of singularity are the solutions of

$$0 = F(r) = r(r-1) + r - 2 = r^2 - 2$$

so $r_1 = \sqrt{2}$ and $r_2 = -\sqrt{2}$.
Assuming $y(x) = \sum_{n=0} a_n x^{r+n}$, differentiating, and substituting into the ODE we obtain

$$\begin{aligned}
0 &= x^2\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{r+n-2} + x\sum_{n=0}^{\infty}(r+n)a_n x^{r+n-1} + (x^2-2)\sum_{n=0}^{\infty} a_n x^{r+n} \\
&= \sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{r+n} + \sum_{n=0}^{\infty}(r+n)a_n x^{r+n} - \sum_{n=0}^{\infty} 2a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} \\
&= \sum_{n=0}^{\infty}\left[(r+n)(r+n-1) + (r+n) - 2\right] a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} \\
&= \sum_{n=0}^{\infty}\left[(r+n)^2 - 2\right] a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} \\
&= \sum_{n=0}^{\infty}\left[(r+n)^2 - 2\right] a_n x^{r+n} + \sum_{n=2}^{\infty} a_{n-2} x^{r+n} \\
&= (r^2-2)a_0 x^r + \left((r+1)^2 - 2\right) a_1 x^{r+1} + \sum_{n=2}^{\infty}\left[(r+n)^2 - 2\right] a_n x^{r+n} + \sum_{n=2}^{\infty} a_{n-2} x^{r+n} \\
&= (r^2-2)a_0 x^r + \left((r+1)^2 - 2\right) a_1 x^{r+1} + \sum_{n=2}^{\infty}\left(\left[(r+n)^2 - 2\right] a_n + a_{n-2}\right) x^{r+n}
\end{aligned}$$

Recurrence relation:

$$a_n(r) = -\frac{a_{n-2}(r)}{(r+n)^2 - 2} \quad \text{for } n \geq 2.$$

When $r = \pm\sqrt{2}$ then the expression

$$\left((r+1)^2 - 2\right) a_1 x^{r+1} = 0$$

which implies $a_1 = 0$. Thus according to the recurrence relation $a_{2n+1} = 0$ for $n \in \mathbb{N}$. If we let $r = r_1 = \sqrt{2}$ and $a_0 = 1$ then the recurrence relation becomes

$$a_n(\sqrt{2}) = -\frac{a_{n-2}(\sqrt{2})}{(\sqrt{2}+n)^2 - 2} = -\frac{a_{n-2}(\sqrt{2})}{n(n+2\sqrt{2})} \quad \text{for } n \geq 2.$$

$$\begin{aligned}
a_2(\sqrt{2}) &= -\frac{1}{2(2+2\sqrt{2})} = -\frac{1}{2^2(1!)(1+\sqrt{2})} \\
a_4(\sqrt{2}) &= -\frac{a_2(\sqrt{2})}{4(4+2\sqrt{2})} = \frac{1}{2^4(2!)(1+\sqrt{2})(2+\sqrt{2})} \\
a_6(\sqrt{2}) &= -\frac{a_4(\sqrt{2})}{6(6+2\sqrt{2})} = -\frac{1}{2^6(3!)(1+\sqrt{2})(2+\sqrt{2})(3+\sqrt{2})} \\
&\vdots \\
a_{2n}(\sqrt{2}) &= \frac{(-1)^n}{2^{2n}(n!)\prod_{k=1}^{n}(k+\sqrt{2})}
\end{aligned}$$

Consequently

$$y_1(x) = x^{\sqrt{2}}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)\prod_{k=1}^{n}(k+\sqrt{2})}\right].$$

If we let $r = r_2 = -\sqrt{2}$ and $a_0 = 1$ then the recurrence relation becomes

$$a_n(-\sqrt{2}) = -\frac{a_{n-2}(-\sqrt{2})}{(-\sqrt{2}+n)^2 - 2} = -\frac{a_{n-2}(-\sqrt{2})}{n(n-2\sqrt{2})} \quad \text{for } n \geq 2.$$

$$\begin{aligned}
a_2(-\sqrt{2}) &= -\frac{1}{2(2-2\sqrt{2})} = -\frac{1}{2^2(1!)(1-\sqrt{2})} \\
a_4(-\sqrt{2}) &= -\frac{a_2(-\sqrt{2})}{4(4-2\sqrt{2})} = \frac{1}{2^4(2!)(1-\sqrt{2})(2-\sqrt{2})} \\
a_6(-\sqrt{2}) &= -\frac{a_4(-\sqrt{2})}{6(6-2\sqrt{2})} = -\frac{1}{2^6(3!)(1-\sqrt{2})(2-\sqrt{2})(3-\sqrt{2})} \\
&\vdots \\
a_{2n}(-\sqrt{2}) &= \frac{(-1)^n}{2^{2n}(n!)\prod_{k=1}^{n}(k-\sqrt{2})}
\end{aligned}$$

Consequently

$$y_2(x) = x^{-\sqrt{2}}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)\prod_{k=1}^{n}(k-\sqrt{2})}\right].$$

The general solution to Bessel's equation of order $\sqrt{2}$ can then be expressed as

$$y(x) = c_1 x^{\sqrt{2}} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!) \prod_{k=1}^{n}(k+\sqrt{2})} \right] + c_2 x^{-\sqrt{2}} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!) \prod_{k=1}^{n}(k-\sqrt{2})} \right].$$