Millersville University **Name** **Answer Key**
Department of Mathematics
MATH 365, *Ordinary Differential Equations*, Homework 10
December 3, 2008

Find the solutions to the following exercises. Answers without justifying work will receive no credit. Partial credit will be given as appropriate, do not leave any problem blank. Each problem is worth 10 points. Your completed assignment is due at class time on Friday, December 5, 2008.

1. Suppose that the fish population is growing exponentially and the local authorities decide to give out fishing licenses that allow a total of h fish per day to be caught over a 30-day period. If $P(t)$ denotes the population of fish in the lake at time t then

$$P'(t) = kP(t) - \begin{cases} h & \text{for } 0 \leq t \leq 30 \\ 0 & \text{for } t > 30. \end{cases}$$

   (a) If $P(0) = P_0$, find $P(t)$.
   Using the unit step function we may re-write the IVP as

$$\begin{aligned} P'(t) &= kP(t) - h + u_{30}(t)h \\ P(0) &= P_0. \end{aligned}$$

   Using the Laplace transform we obtain

$$\begin{aligned} \mathcal{L}\{P'(t)\} &= \mathcal{L}\{kP(t)\} - \mathcal{L}\{h\} + \mathcal{L}\{u_{30}(t)h\} \\ sp(s) - P(0) &= kp(s) - \frac{h}{s} + e^{-30s}\frac{h}{s} \\ (s-k)p(s) &= P_0 - \frac{h}{s} + e^{-30s}\frac{h}{s} \\ p(s) &= \frac{P_0}{s-k} - \frac{h}{s(s-k)} + e^{-30s}\frac{h}{s(s-k)} \\ &= \frac{P_0}{s-k} + \frac{h/k}{s} - \frac{h/k}{s-k} + e^{-30s}\left(\frac{h/k}{s-k} - \frac{h/k}{s}\right) \\ P(t) &= \left(P_0 - \frac{h}{k}\right)e^{kt} + \frac{h}{k} + \frac{h}{k}u_{30}(t)\left(e^{k(t-30)} - 1\right) \end{aligned}$$

(b) Find a relationship between P_0, h, and k so that $P(360) = P_0$.

$$
\begin{aligned}
P_0 &= \left(P_0 - \frac{h}{k}\right) e^{360k} + \frac{h}{k} + \frac{h}{k} u_{30}(360)\left(e^{k(360-30)} - 1\right) \\
&= \left(P_0 - \frac{h}{k}\right) e^{360k} + \frac{h}{k} + \frac{h}{k} e^{330k} - \frac{h}{k} \\
&= \left(P_0 - \frac{h}{k}\right) e^{360k} + \frac{h}{k} e^{330k}
\end{aligned}
$$

2. A wooden beam with ends at $x = 0$ and $x = L$ will bend when a load of $W(x)$ is placed on the beam. The amount of bending is modeled by the IVP:

$$\begin{aligned} y^{(4)} &= W(x) = \begin{cases} W_0 & \text{for } 0 < x < L/2 \\ 0 & \text{for } L/2 < x < L. \end{cases} \\ y(0) &= 0 \\ y'(0) &= 0 \\ y''(0) &= 0 \\ y'''(0) &= 0 \end{aligned}$$

Solve the IVP for $y(x)$.

Using the unit step function we can re-write the IVP in the form:

$$\begin{aligned} y^{(4)} &= W_0 - u_{L/2}(x)W_0 \\ y(0) &= 0 \\ y'(0) &= 0 \\ y''(0) &= 0 \\ y'''(0) &= 0. \end{aligned}$$

Applying the Laplace transform we obtain

$$\begin{aligned} \mathcal{L}\left\{y^{(4)}\right\} &= \mathcal{L}\left\{W_0\right\} - \mathcal{L}\left\{u_{L/2}(x)W_0\right\} \\ s^4Y(s) &= \frac{W_0}{s} - e^{-(L/2)s}\frac{W_0}{s} \\ Y(s) &= \frac{W_0}{s^5} - e^{-(L/2)s}\frac{W_0}{s^5} \\ y(x) &= \frac{W_0}{4!}\left[x^4 - u_{L/2}(x)(x - L/2)^4\right] \end{aligned}$$

3. Evaluate $\mathcal{L}\left\{\delta(t-\pi)\cos t^3\right\}$ where $\delta(t)$ denotes the Dirac delta function.

$$\begin{aligned}\mathcal{L}\left\{\delta(t-\pi)\cos t^3\right\} &= \int_0^\infty e^{-st}\,\delta(t-\pi)\cos t^3\,dt \\ &= \int_{-\infty}^\infty e^{-st}\,\delta(t-\pi)\cos t^3\,dt \\ &= e^{-\pi s}\cos\pi^3\end{aligned}$$

4. Evaluate the following improper integral.

$$\int_{-\infty}^\infty e^{(t-1)^2}\,\delta(3t)\,dt$$

If we make the substitution $u = 3t$ and $\frac{1}{3}\,du = dt$ then

$$\begin{aligned}\int_{-\infty}^\infty e^{(t-1)^2}\,\delta(3t)\,dt &= \frac{1}{3}\int_{-\infty}^\infty e^{(u/3-1)^2}\,\delta(u)\,du \\ &= \frac{1}{3}e^{(0/3-1)^2} \\ &= \frac{e}{3}\end{aligned}$$

5. Evaluate the following improper integral.

$$\int_{-\infty}^{\infty} \delta'(t) f(t)\, dt$$

We will integrate by parts choosing

$$\begin{aligned} u &= f(t) & \qquad v &= \delta(t) \\ du &= f'(t)\, dt & \qquad dv &= \delta'(t)\, dt. \end{aligned}$$

Thus we have

$$\begin{aligned} \int_{-\infty}^{\infty} \delta'(t) f(t)\, dt &= -\int_{-\infty}^{\infty} \delta(t) f'(t)\, dt \\ &= -f'(0). \end{aligned}$$