Millersville University **Name** ____________________
Department of Mathematics
MATH 365, *Ordinary Differential Equations*, Homework 02
January 21, 2009

Please answer the following questions. Answers without justifying work will receive no credit. Partial credit will be given as appropriate, do not leave any problem blank. Each problem is worth 10 points. Your completed assignment is due at class time on Friday, January 23, 2009.

1. Find the solution to the following ordinary differential equation.

$$y' + 2t\,y = te^{-t^2}$$

This is a first-order linear ODE. The appropriate integrating factor is

$$\mu(t) = e^{\int 2t\,dt} = e^{t^2}.$$

Multiplying both sides of the ODE by the integrating factor yields:

$$\begin{aligned} e^{t^2}\left[y' + 2t\,y\right] &= e^{t^2}te^{-t^2} \\ \frac{d}{dt}\left[ye^{t^2}\right] &= t \\ y(t)e^{t^2} - y(0) &= \frac{1}{2}t^2 \\ y(t) &= y(0)e^{-t^2} + \frac{1}{2}t^2e^{-t^2}. \end{aligned}$$

2. Consider the ordinary differential equation

$$y' + (\cos t)y = e^{-\sin t}.$$

Find the solution $\phi(t)$ which satisfies the condition $\phi(\pi) = \pi$.

This is a first-order linear ODE. The appropriate integrating factor is

$$\mu(t) = e^{\int_\pi^t \cos s\,ds} = e^{\sin t}.$$

Multiplying both sides of the ODE by the integrating factor yields:

$$\begin{aligned} e^{\sin t}\left[y' + (\cos t)y\right] &= e^{\sin t}e^{-\sin t} \\ \frac{d}{dt}\left[ye^{\sin t}\right] &= 1 \\ y(t)e^{\sin t} - y(\pi) &= \int_\pi^t 1\,ds \\ y(t)e^{\sin t} - \pi &= t - \pi \\ y(t) &= te^{-\sin t}. \end{aligned}$$

3. The ordinary differential equation,

$$y' + p(t)\,y = g(t)y^k$$

where k is a constant, is called **Bernoulli's equation**.

(a) Show that if we make the substitution $z = y^{1-k}$, Bernoulli's equation becomes the linear equation,

$$z' + (1-k)p(t)\,z = (1-k)g(t).$$

If $z = y^{1-k}$ then

$$\begin{aligned} \frac{dz}{dt} &= (1-k)y^{-k}\frac{dy}{dt} \\ \frac{y^k}{1-k}\frac{dz}{dt} &= \frac{dy}{dt}. \end{aligned}$$

Thus Bernoulli's equation becomes

$$\begin{aligned} y' + p(t)\,y &= g(t)y^k \\ \frac{y^k}{1-k}\frac{dz}{dt} + p(t)\,y &= g(t)y^k \\ \frac{1}{1-k}\frac{dz}{dt} + p(t)\,y^{1-k} &= g(t) \\ \frac{dz}{dt} + (1-k)p(t)\,z &= (1-k)g(t). \end{aligned}$$

(b) Use this substitution technique to solve the ordinary differential equation

$$y' - 2t\,y = t\,y^2.$$

In this case $k = 2$ so $z = y^{-1}$ and the transformed equation takes on the form

$$z' + 2t\,z = -t.$$

The integrating factor for this first-order linear equation is

$$\mu(t) = e^{\int 2t\,dt} = e^{t^2}.$$

Multiplying both sides of the ODE by the integrating factor yields:

$$\begin{aligned} e^{t^2}\left[z' + 2t\,z\right] &= -t\,e^{t^2} \\ \frac{d}{dt}\left[ze^{t^2}\right] &= -t\,e^{t^2} \\ z(t)e^{t^2} - z(0) &= -\frac{1}{2}e^{t^2} \\ z(t) &= z(0)e^{-t^2} - \frac{1}{2}. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{1}{y(t)} &= \frac{1}{y(0)} e^{-t^2} - \frac{1}{2} \\ y(t) &= \frac{1}{\frac{1}{y(0)} e^{-t^2} - \frac{1}{2}}. \end{aligned}$$

4. Suppose $\phi(t)$ is a function with a continuous first derivative and for which $\phi(0) = 1$. Furthermore suppose $\phi'(t) - 2\phi(t) \leq 1$. Show that $\phi(t) \leq \frac{3}{2}e^{2t} - \frac{1}{2}$. (*Hint:* use an integrating factor and properties of the definite integral on the differential inequality given.)

   An integrating factor for this inequality would be

$$\mu(t) = e^{\int_0^t (-2)\, ds} = e^{-2t}.$$

   Multiplying both sides of the equality by the integrating factor yields

$$\begin{aligned} e^{-2t}\left[\phi'(t) - 2\phi(t)\right] &\leq e^{-2t}\left[1\right] \\ \frac{d}{dt}\left[\phi(t)e^{-2t}\right] &\leq e^{-2t} \\ \phi(t)e^{-2t} - \phi(0) &\leq \int_0^t e^{-2s}\, ds \\ \phi(t)e^{-2t} - 1 &\leq -\frac{1}{2}e^{-2t} + \frac{1}{2} \\ \phi(t)e^{-2t} &\leq -\frac{1}{2}e^{-2t} + \frac{3}{2} \\ \phi(t) &\leq \frac{3}{2}e^{2t} - \frac{1}{2}. \end{aligned}$$