Millersville University **Name** <u>**Answer Key**</u>
Department of Mathematics
MATH 365, *Ordinary Differential Equations*, Homework 03
January 28, 2009

Please answer the following questions. Answers without justifying work will receive no credit. Partial credit will be given as appropriate, do not leave any problem blank. Each problem is worth 10 points. Your completed assignment is due at class time on Monday, February 2, 2009.

1. A liquid carries a drug into an organ of volume 500 cm^3 at a rate of 10 cm^3/sec and leaves at the same rate. The concentration of drug in the entering liquid is 0.08 g/cm^3. Assuming that the drug is not present in the organ initially, find

   (a) the concentration of the drug in the organ after 30 seconds,

   If $Q(t)$ represents the amount of drug in the organ as a function of time t, then the rate of change in Q is the difference between the inflow and the outflow.

$$\begin{aligned}\frac{dQ}{dt} &= (0.08)(10) - \frac{Q}{V}(10)\\ &= 0.8 - \frac{10Q}{500} = 0.8 - \frac{1}{50}Q\end{aligned}$$

   Since concentration $C(t) = Q(t)/V$ then

$$\begin{aligned}\frac{1}{V}\frac{dQ}{dT} &= \frac{1}{500}\left(0.8 - \frac{1}{50}Q\right)\\ \frac{dC}{dt} &= 0.0016 - \frac{1}{50}C.\end{aligned}$$

   The initial concentration is $C(0) = 0$ g/cm^3.

   We can solve this first-order linear IVP my multiplying the equation by an integrating factor.

$$\begin{aligned}\frac{dC}{dt} + \frac{1}{50}C &= 0.0016\\ e^{t/50}\left[\frac{dC}{dt} + \frac{1}{50}C\right] &= 0.0016e^{t/50}\\ \frac{d}{dt}\left[e^{t/50}C\right] &= 0.0016e^{t/50}\\ e^{t/50}C(t) &= 0.08(e^{t/50} - 1)\\ C(t) &= 0.08(1 - e^{-t/50})\end{aligned}$$

   At $t = 30$ the concentration of drug in the organ will be

$$C(30) = 0.08(1 - e^{-30/50}) \approx 0.036095 \quad \text{g/cm}^3.$$

(b) the equilibrium concentration of drug in the organ,

When the drug reaches equilibrium concentration in the organ, $C'(t) = 0$, thus

$$\begin{aligned}
\frac{dC}{dt} + \frac{1}{50}C &= 0.0016 \\
\frac{1}{50}C &= 0.0016 \\
C &= 0.08 \quad \text{g/cm}^3.
\end{aligned}$$

(c) when does the concentration of drug in the organ reach 0.06 g/cm^3?

$$\begin{aligned}
0.06 &= 0.08(1 - e^{-t/50}) \\
0.75 &= 1 - e^{-t/50} \\
-0.25 &= -e^{-t/50} \\
0.25 &= e^{-t/50} \\
\ln 0.25 &= -\frac{t}{50} \\
\ln 4 &= \frac{t}{50} \\
t &= 50 \ln 4 \\
&\approx 69.3147 \quad \text{sec.}
\end{aligned}$$

2. Suppose that the maximum concentration of a drug present in a given organ of constant volume V must be $c_{\max}$. Assuming the organ does not contain the drug initially, the the liquid carrying the drug into the organ has constant concentration $c > c_{\max}$, and that the inflow and outflow rates are equal to r, find the maximum length of time the liquid can be allowed to flow into the organ.

   If $Q(t)$ represents the amount of drug in the organ as a function of time t, then the rate of change in Q is the difference between the inflow and the outflow.

$$\frac{dQ}{dt} = cr - r\frac{Q}{V}$$

   Since concentration $C(t) = Q(t)/V$ then

$$\begin{aligned}
\frac{1}{V}\frac{dQ}{dT} &= \frac{1}{V}\left(cr - r\frac{Q}{V}\right) \\
\frac{dC}{dt} &= \frac{cr}{V} - \frac{r}{V}C.
\end{aligned}$$

The initial concentration is $C(0) = 0$.

We can solve this first-order linear IVP my multiplying the equation by an integrating factor.

$$\begin{aligned} \frac{dC}{dt} + \frac{r}{V}C &= \frac{cr}{V} \\ e^{rt/V}\left[\frac{dC}{dt} + \frac{r}{V}C\right] &= \frac{cr}{V}e^{rt/V} \\ \frac{d}{dt}\left[e^{rt/V}C\right] &= \frac{cr}{V}e^{rt/V} \\ e^{rt/V}C(t) &= c(e^{rt/V} - 1) \\ C(t) &= c(1 - e^{-rt/V}) \end{aligned}$$

$C(t) = c_{\max}$ when the following equation in true.

$$\begin{aligned} c_{\max} &= c(1 - e^{-rt/V}) \\ \frac{c_{\max}}{c} &= 1 - e^{-rt/V} \\ e^{-rt/V} &= 1 - \frac{c_{\max}}{c} \\ -\frac{rt}{V} &= \ln\left(1 - \frac{c_{\max}}{c}\right) \\ \frac{rt}{V} &= \ln\frac{1}{1 - \frac{c_{\max}}{c}} \\ &= \ln\frac{c}{c - c_{\max}} \\ t &= \frac{V}{r}\ln\frac{c}{c - c_{\max}} \end{aligned}$$

3. Suppose that a liquid carries a drug into an organ at a rate r_{in} and that liquid flows away from the organ at rate $r_{out} < r_{in}$. Thus the volume of the organ at time $t \geq 0$ is given by $V = V_0 + rt$. The concentration of drug in the incoming liquid is constant c. The concentration of drug in the outflowing liquid varies with t and equals the well-mixed concentration of drug in the organ. Suppose the initial concentration of drug in the organ is C_0. Find the concentration $C(t)$ of drug in the organ at time $t > 0$.

   If $Q(t)$ represents the amount of drug in the organ as a function of time t, then the rate of change in Q is the difference between the inflow and the outflow.

$$\begin{aligned} \frac{dQ}{dt} &= cr_{in} - r_{out}\frac{Q}{V} \\ &= cr_{in} - r_{out}\frac{Q}{V_0 + rt} \end{aligned}$$

Since concentration $C(t) = Q(t)/(V_0 + rt)$ then according to the quotient rule for derivatives

$$\begin{aligned}\frac{dC}{dt} &= \frac{d}{dt}\left[\frac{Q(t)}{V_0 + rt}\right] \\ &= \frac{\frac{dQ}{dt}(V_0 + rt) - Q(t)r}{(V_0 + rt)^2} \\ &= \frac{1}{V_0 + rt}\frac{dQ}{dt} - \frac{r}{V_0 + rt}\frac{Q(t)}{V_0 + rt} \\ &= \frac{1}{V_0 + rt}\frac{dQ}{dt} - \frac{r}{V_0 + rt}C(t).\end{aligned}$$

Dividing both sides of the differential equation by the volume and using the derivative calculated above yields the equation:

$$\begin{aligned}\frac{1}{V_0 + rt}\frac{dQ}{dt} &= \frac{1}{V_0 + rt}\left(cr_{in} - r_{out}\frac{Q}{V_0 + rt}\right) \\ \frac{dC}{dt} + \frac{r}{V_0 + rt}C(t) &= \frac{1}{V_0 + rt}\left(cr_{in} - C(t)r_{out}\right) \\ \frac{dC}{dt} + \frac{r + r_{out}}{V_0 + rt}C(t) &= \frac{cr_{in}}{V_0 + rt}.\end{aligned}$$

The initial concentration is $C(0) = C_0$.

We can solve this first-order linear IVP my multiplying the equation by an integrating factor. In this case the integrating factor is

$$\begin{aligned}\mu(t) &= e^{\int \frac{r + r_{out}}{V_0 + rt}\,dt} \\ &= e^{(1 + r_{out}/r)\ln(V_0 + rt)} \\ &= (V_0 + rt)^{(1 + r_{out}/r)}.\end{aligned}$$

Solving the IVP produces

$$\begin{aligned}(V_0 + rt)^{(1 + r_{out}/r)}\left[\frac{dC}{dt} + \frac{r + r_{out}}{V_0 + rt}C(t)\right] &= (V_0 + rt)^{(1 + r_{out}/r)}\left[\frac{cr_{in}}{V_0 + rt}\right] \\ \frac{d}{dt}\left[C(t)(V_0 + rt)^{(1 + r_{out}/r)}\right] &= cr_{in}(V_0 + rt)^{r_{out}/r} \\ C(t)(V_0 + rt)^{(1 + r_{out}/r)} - C_0 V_0^{(1 + r_{out}/r)} &= cr_{in}\int_0^t (V_0 + rs)^{r_{out}/r}\,ds \\ &= \frac{cr_{in}}{r + r_{out}}\left[(V_0 + rt)^{(1 + r_{out}/r)} - V_0^{(1 + r_{out}/r)}\right] \\ C(t)(V_0 + rt)^{(1 + r_{out}/r)} &= C_0 V_0^{(1 + r_{out}/r)} \\ &\quad + \frac{cr_{in}}{r + r_{out}}\left[(V_0 + rt)^{(1 + r_{out}/r)} - V_0^{(1 + r_{out}/r)}\right]\end{aligned}$$

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\begin{aligned}
C(t) &= C_0 \left(\frac{V_0}{V_0 + rt} \right)^{(1 + r_{out}/r)} \\
&\quad + \frac{c r_{in}}{r + r_{out}} \left[1 - \left(\frac{V_0}{V_0 + rt} \right)^{(1 + r_{out}/r)} \right] \\
&= \left(C_0 - \frac{c r_{in}}{r + r_{out}} \right) \left(\frac{V_0}{V_0 + rt} \right)^{(1 + r_{out}/r)} + \frac{c r_{in}}{r + r_{out}}
\end{aligned}
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