

Please answer the following questions. Answers without justifying work will receive no credit. Partial credit will be given as appropriate, do not leave any problem blank. Each problem is worth 10 points. Your completed assignment is due at class time on Monday, February 9, 2009.

1. Show that the following first-order ordinary differential equations are exact and find their solutions in implicit form.

(a) $\cos t \cos^2 y \, dt - \sin t \sin 2y \, dy = 0$

The ODE is exact since

$$\begin{aligned}\frac{\partial}{\partial y} (\cos t \cos^2 y) &= \cos t (-2 \cos y \sin y) \\ &= -\cos t \sin 2y \\ &= \frac{\partial}{\partial t} (-\sin t \sin 2y).\end{aligned}$$

The solution in implicit form can be written as

$$\begin{aligned}\psi(t, y) &= \int \cos t \cos^2 y \, dt \\ &= \sin t \cos^2 y + h(y)\end{aligned}$$

where $h(y)$ is an arbitrary function of y .

$$\begin{aligned}\frac{\partial}{\partial y} \psi(t, y) &= \frac{\partial}{\partial y} (\sin t \cos^2 y + h(y)) \\ -\sin t \sin 2y &= -\sin t \sin 2y + h'(y) \\ 0 &= h'(y)\end{aligned}$$

Thus $h(y)$ is a constant. Consequently we may write the solution to the ODE in implicit form as

$$\psi(t, y) = \sin t \cos^2 y = C.$$

(b) $(2y e^{2t} + 2t \cos y) \, dt + (e^{2t} - t^2 \sin y) \, dy = 0$

The ODE is exact since

$$\begin{aligned}\frac{\partial}{\partial y} (2y e^{2t} + 2t \cos y) &= 2e^{2t} - 2t \sin y \\ &= \frac{\partial}{\partial t} (e^{2t} - t^2 \sin y).\end{aligned}$$

The solution in implicit form can be written as

$$\begin{aligned}\psi(t, y) &= \int 2y e^{2t} + 2t \cos y \, dt \\ &= y e^{2t} + t^2 \cos y + h(y)\end{aligned}$$

where $h(y)$ is an arbitrary function of y .

$$\begin{aligned}\frac{\partial}{\partial y} \psi(t, y) &= \frac{\partial}{\partial y} (y e^{2t} + t^2 \cos y + h(y)) \\ e^{2t} - t^2 \sin y &= e^{2t} - t^2 \sin y + h'(y) \\ 0 &= h'(y)\end{aligned}$$

Thus $h(y)$ is a constant. Consequently we may write the solution to the ODE in implicit form as

$$\psi(t, y) = y e^{2t} + t^2 \cos y = C.$$

2. Show that the following first-order ordinary differential equations are not exact, find integrating factors which make them exact, and find their solutions in implicit form.

(a) $\cos t \cos y \, dt - 2 \sin t \sin y \, dy = 0$

Taking partial derivatives we see that

$$\begin{aligned}\frac{\partial}{\partial y} (\cos t \cos y) &= -\cos t \sin y \\ &\neq -2 \cos t \sin y \\ &= \frac{\partial}{\partial t} (-2 \sin t \sin y)\end{aligned}$$

and therefore the ODE is not exact. We can look for an integrating factor of the form $\mu(t)$.

$$\begin{aligned}\frac{\partial}{\partial y} \mu(t) (\cos t \cos y) &= \frac{\partial}{\partial t} \mu(t) (-2 \sin t \sin y) \\ -\mu(t) \cos t \sin y &= -2\mu(t) \cos t \sin y - \mu'(t) (-2 \sin t \sin y) \\ \mu(t) \cos t \sin y &= -2\mu'(t) \sin t \sin y \\ \mu(t) \cos t \sin y + 2\mu'(t) \sin t \sin y &= 0 \\ \mu'(t) + \frac{\cot t}{2} \mu(t) &= 0 \\ \sqrt{\sin t} \left[\mu'(t) + \frac{\cot t}{2} \mu(t) \right] &= 0 \\ \frac{d}{dt} \left[\sqrt{\sin t} \mu(t) \right] &= 0 \\ \sqrt{\sin t} \mu(t) &= 1 \\ \mu(t) &= \sqrt{\csc t}\end{aligned}$$

Now we can see that

$$\begin{aligned}
 \frac{\partial}{\partial y} \sqrt{\csc t} (\cos t \cos y) &= -\sqrt{\csc t} \cos t \sin y \\
 &= \sqrt{\csc t} \cos t \sin y - 2\sqrt{\csc t} \cos t \sin y \\
 &= \frac{\csc t \cot t}{\sqrt{\csc t}} (\sin t \sin y) - 2\sqrt{\csc t} \cos t \sin y \\
 &= -\frac{\csc t \cot t}{2\sqrt{\csc t}} (-2 \sin t \sin y) + \sqrt{\csc t} (-2 \cos t \sin y) \\
 &= \frac{\partial}{\partial t} \sqrt{\csc t} (-2 \sin t \sin y)
 \end{aligned}$$

and the ODE is now exact. We can write the solution in implicit form as

$$\begin{aligned}
 \psi(t, y) &= -2 \int \sqrt{\csc t} \sin t \sin y \, dy \\
 &= 2\sqrt{\sin t} \cos y + h(t)
 \end{aligned}$$

where $h(t)$ is an arbitrary function of t .

$$\begin{aligned}
 \frac{\partial}{\partial t} \psi(t, y) &= \frac{\partial}{\partial t} (2\sqrt{\sin t} \cos y + h(t)) \\
 \sqrt{\csc t} \cos t \cos y &= \sqrt{\csc t} \cos t \cos y + h'(t) \\
 0 &= h'(t)
 \end{aligned}$$

Thus $h(t)$ is a constant. Consequently we may write the solution to the ODE in implicit form as

$$\psi(t, y) = 2\sqrt{\sin t} \cos y = C.$$

(b) $(e^t + t e^y) dt + t e^y dy = 0$

Taking partial derivatives we see that

$$\begin{aligned}
 \frac{\partial}{\partial y} (e^t + t e^y) &= t e^y \\
 &\neq e^y \\
 &= \frac{\partial}{\partial t} (t e^y)
 \end{aligned}$$

and therefore the ODE is not exact. We will look for an integrating factor of the form $\mu(t)$.

$$\frac{\partial}{\partial y} \mu(t) (e^t + t e^y) = \frac{\partial}{\partial t} \mu(t) (t e^y)$$

$$\begin{aligned}
\mu(t)t e^y &= \mu'(t)t e^y + \mu(t)e^y \\
\mu(t)t &= \mu'(t)t + \mu(t) \\
(t-1)\mu(t) &= \mu'(t)t \\
\mu'(t) + \left(\frac{1}{t} - 1\right)\mu(t) &= 0 \\
te^{-t} \left[\mu'(t) + \left(\frac{1}{t} - 1\right)\mu(t) \right] &= 0 \\
\frac{d}{dt} [te^{-t}\mu(t)] &= 0 \\
te^{-t}\mu(t) &= 1 \\
\mu(t) &= \frac{1}{t}e^t
\end{aligned}$$

We can now check that after multiplying by the integrating factor, the ODE is exact.

$$\begin{aligned}
\frac{\partial}{\partial y} \frac{1}{t} e^t (e^t + t e^y) &= e^t e^y \\
&= \frac{\partial}{\partial t} (e^t e^y) \\
&= \frac{\partial}{\partial t} \frac{1}{t} e^t (t e^y)
\end{aligned}$$

We can write the solution in implicit form as

$$\begin{aligned}
\psi(t, y) &= \int \frac{1}{t} e^t (t e^y) dy \\
&= \int e^t e^y dy \\
&= e^t e^y + h(t)
\end{aligned}$$

where $h(t)$ is an arbitrary function of t .

$$\begin{aligned}
\frac{\partial}{\partial t} \psi(t, y) &= \frac{\partial}{\partial t} (e^t e^y + h(t)) \\
\frac{1}{t} e^t (e^t + t e^y) &= e^t e^y + h'(t) \\
\frac{1}{t} e^{2t} + e^t e^y &= e^t e^y + h'(t) \\
\frac{1}{t} e^{2t} &= h'(t) \\
h(t) &= \int_1^t \frac{1}{s} e^{2s} ds
\end{aligned}$$

Consequently we may write the solution to the ODE in implicit form as

$$\psi(t, y) = e^t e^y + \int_1^t \frac{1}{s} e^{2s} ds = C.$$

3. Calculate the first four terms of the sequence of successive approximations to the initial value problem

$$\begin{aligned}\frac{dy}{dt} &= t^2 + y^2 \\ y(0) &= 0.\end{aligned}$$

If $\phi_0(t) = 0$ then

$$\begin{aligned}\phi_1(t) &= \int_0^t (s^2 + 0^2) ds \\ &= \int_0^t s^2 ds \\ &= \frac{1}{3}t^3\end{aligned}$$

$$\begin{aligned}\phi_2(t) &= \int_0^t \left(s^2 + \left[\frac{1}{3}s^3 \right]^2 \right) ds \\ &= \int_0^t \left(s^2 + \frac{1}{9}s^6 \right) ds \\ &= \frac{1}{3}t^3 + \frac{1}{63}t^7\end{aligned}$$

$$\begin{aligned}\phi_3(t) &= \int_0^t \left(s^2 + \left[\frac{1}{3}s^3 + \frac{1}{63}s^7 \right]^2 \right) ds \\ &= \int_0^t \left(s^2 + \frac{1}{9}s^6 + \frac{2}{189}s^{10} + \frac{1}{3969}s^{14} \right) ds \\ &= \frac{1}{3}t^3 + \frac{1}{63}t^7 + \frac{2}{2079}t^{11} + \frac{1}{59535}t^{15}\end{aligned}$$