Millersville University **Name** **Answer Key**
Department of Mathematics
MATH 365, *Ordinary Differential Equations*, Homework 05
February 18, 2009

Please answer the following questions. Answers without justifying work will receive no credit. Partial credit will be given as appropriate, do not leave any problem blank. Each problem is worth 10 points. Your completed assignment is due at class time on Friday, February 20, 2009.

1. Find the solution to the following initial value problem.

$$\begin{aligned} y'' - 2y' - 3y &= 0 \\ y(0) &= 0 \\ y'(0) &= 1 \end{aligned}$$

The characteristic equation corresponding to this second-order linear, constant coefficient, homogeneous equation is

$$0 = r^2 - 2r - 3 = (r-3)(r+1)$$

which implies $r_1 = -1$ and $r_2 = 3$. Thus the general solution of the equation is

$$y(t) = c_1 e^{-t} + c_2 e^{3t}.$$

Applying the initial conditions we obtain the set of linear equations below.

$$\begin{aligned} 0 &= y(0) = c_1 + c_2 \\ 1 &= y'(0) = -c_1 + 3c_2 \end{aligned}$$

Solving this system produces $c_1 = -1/4$ and $c_2 = 1/4$. Thus the solution to the initial value problem is

$$y(t) = -\frac{1}{4}e^{-t} + \frac{1}{4}e^{3t}$$

2. Let $\phi_1(t) = \cos t$ and $\phi_2(t) = \sin t$. Determine whether $\phi_1(t)$ and $\phi_2(t)$ are linearly independent on the interval $(-\infty, \infty)$.

$$
\begin{aligned}
W(\cos t, \sin t)(t) &= \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} \\
&= \cos^2 t + \sin^2 t \\
&= 1 \neq 0
\end{aligned}
$$

Thus $\phi_1(t)$ and $\phi_2(t)$ are linearly independent on the interval $(-\infty, \infty)$.

3. Use Abel's Theorem to determine the Wronskian of two solutions to the ordinary differential equation

$$t^2 y'' + ty' + (t^2 - n^2)y = 0$$

where n is an arbitrary integer.

Re-writing the ordinary differential equation in the standard form

$$y'' + \frac{1}{t}y' + \left(1 - \frac{n^2}{t^2}\right) y = 0$$

we can assign $p(t) = 1/t$. According to Abel's Theorem

$$W = Ce^{-\int p(t)\,dt} = Ce^{-\int (1/t)\,dt} = Ce^{-\ln t} = \frac{C}{t}$$

where C is a constant.

4. Consider the ordinary differential equation

$$(1-t^2)y'' - 2ty' + 2y = 0, \quad \text{for } 0 < t < 1.$$

Verify that $\phi_1(t) = t$ is a solution to this equation. Use the Wronskian given by Abel's Theorem to find a second, linearly independent solution to this equation.

Substituting the solution into the differential equation yields

$$(1-t^2)(t)'' - 2t(t)' + 2t = 0 - 2t + 2t = 0$$

thus $\phi_1(t) = t$ is a solution to the ordinary differential equation. Re-writing the ordinary differential equation in the standard form

$$y'' - \frac{2t}{1-t^2}y' + \frac{2}{1-t^2}y = 0$$

we can assign $p(t) = -2t/(1-t^2)$. According to Abel's Theorem

$$W = Ce^{-\int p(t)\,dt} = Ce^{-\int (-2t/(1-t^2))\,dt} = Ce^{-\ln|1-t^2|} = \frac{C}{1-t^2}.$$

Let the second, linearly independent solution be $\phi_2(t)$. If we compute the Wronskian of the two solutions directly we have

$$W = \begin{vmatrix} t & \phi_2(t) \\ 1 & \phi_2'(t) \end{vmatrix} = t\phi_2'(t) - \phi_2(t).$$

Equating the two forms of the Wronskian produces the first order linear differential equation below.

$$\begin{aligned} t\phi_2'(t) - \phi_2(t) &= \frac{C}{|1-t^2|} \\ \phi_2'(t) - \frac{1}{t}\phi_2(t) &= \frac{C}{t(1-t^2)} \end{aligned}$$

An integrating factor for this ODE is $\mu(t) = 1/t$.

$$\begin{aligned} \frac{1}{t}\left[\phi_2'(t) - \frac{1}{t}\phi_2(t)\right] &= \frac{C}{t^2(1-t^2)} \\ \frac{d}{dt}\left[\frac{1}{t}\phi_2(t)\right] &= C\left(\frac{1}{t^2} + \frac{1}{2}\frac{1}{1-t} + \frac{1}{2}\frac{1}{1+t}\right) \quad \text{(partial fractions)} \\ \frac{1}{t}\phi_2(t) &= C\left(-\frac{1}{t} - \frac{1}{2}\ln(1-t) + \frac{1}{2}\ln(1+t)\right) + \hat{C} \\ \phi_2(t) &= C\left(-1 + t\ln\sqrt{\frac{1+t}{1-t}}\right) + \hat{C}t \end{aligned}$$

5. Let $\phi_1(t) = t^2$ and $\phi_2(t) = t|t|$. Determine whether $\phi_1(t)$ and $\phi_2(t)$ are linearly independent on the interval $[-1, 1]$.

Suppose there are constants k_1 and k_2 such that

$$k_1\phi_1(t) + k_2\phi_2(t) = 0$$

for all $-1 \leq t \leq 1$. Then

$$\begin{aligned} 0 &= k_1t^2 + k_2t|t| \\ &= k_1t^2 + k_2\begin{cases} -t^2 & \text{if } t < 0 \\ t^2 & \text{if } t \geq 0 \end{cases} \\ &= k_1t^2 + \begin{cases} -k_2t^2 & \text{if } t < 0 \\ k_2t^2 & \text{if } t \geq 0 \end{cases} \\ &= \begin{cases} (k_1 - k_2)t^2 & \text{if } t < 0 \\ (k_1 + k_2)t^2 & \text{if } t \geq 0 \end{cases} \end{aligned}$$

which implies

$$\begin{aligned} k_1 - k_2 &= 0 \\ k_1 + k_2 &= 0 \end{aligned}$$

and consequently $k_1 = k_2 = 0$. Thus the functions are linearly independent on $[-1, 1]$.