Millersville University **Name** __________Answer Key__________
Department of Mathematics
MATH 365, *Ordinary Differential Equations*, Homework 06
February 25, 2009

Please answer the following questions. Answers without justifying work will receive no credit. Partial credit will be given as appropriate, do not leave any problem blank. Each problem is worth 10 points. Your completed assignment is due at class time on Friday, February 27, 2009.

1. Find the general solutions to the following ordinary differential equations.

   (a) $4y'' - y = e^t$

   First we will find the complementary solution $y_c(t)$ to the ODE:

   $$4y'' - y = 0.$$

   The characteristic equation has the form

   $$4r^2 - 1 = 0$$

   which implies $r_1 = -\frac{1}{2}$ and $r_2 = \frac{1}{2}$. Therefore

   $$y_c(t) = c_1 e^{-t/2} + c_2 e^{t/2}.$$

   Assuming the particular solution has the form $Y(t) = Ae^t$, upon differentiating the particular solution and substituting into the nonhomogeneous equation we obtain

   $$\begin{aligned} 4Ae^t - Ae^t &= e^t \\ 3Ae^t &= e^t \\ 3A &= 1 \\ A &= \frac{1}{3}. \end{aligned}$$

   Consequently the general solution to the nonhomogeneous differential equation is

   $$y(t) = y_c(t) + Y(t) = c_1 e^{-t/2} + c_2 e^{t/2} + \frac{1}{3}e^t.$$

(b) $y'' - 7y' + 6y = \sin t$

First we will find the complementary solution $y_c(t)$ to the ODE:

$$y'' - 7y' + 6y = 0.$$

The characteristic equation has the form

$$\begin{aligned} r^2 - 7r + 6 &= 0 \\ (r-1)(r-6) &= 0 \end{aligned}$$

which implies $r_1 = 1$ and $r_2 = 6$. Therefore

$$y_c(t) = c_1 e^t + c_2 e^{6t}.$$

Assuming the particular solution has the form $Y(t) = A\sin t + B\cos t$, upon differentiating the particular solution and substituting into the nonhomogeneous equation we obtain

$$\begin{aligned} -A\sin t - B\cos t - 7(A\cos t - B\sin t) + 6(A\sin t + B\cos t) &= \sin t \\ (-A + 7B + 6A)\sin t + (-B - 7A + 6B)\cos t &= \sin t \\ (5A + 7B)\sin t + (-7A + 5B)\cos t &= \sin t \end{aligned}$$

Thus we must solve the system of linear equations

$$\begin{aligned} 5A + 7B &= 1 \\ -7A + 5B &= 0. \end{aligned}$$

Multiplying the first equation by 7 and the second equation by 5 and adding produces $B = 7/74$ which implies $A = 5/74$. Consequently the general solution to the nonhomogeneous differential equation is

$$y(t) = y_c(t) + Y(t) = c_1 e^t + c_2 e^{6t} + \frac{5}{74}\sin t + \frac{7}{74}\cos t.$$

(c) $y'' + 16y = 2\sin t\,\sin 2t$ (*Hint*: use a product-to-sum formula.)

First we will find the complementary solution $y_c(t)$ to the ODE:

$$y'' + 16y = 0.$$

The characteristic equation has the form

$$r^2 + 16 = 0$$

which implies $r = 0 \pm 4i$. Therefore

$$y_c(t) = c_1 \cos 4t + c_2 \sin 4t.$$

Before solving for the particular solution we will re-write the nonhomogeneous part of the ODE using the product-to-sum formula

$$2\sin t\,\sin 2t = \cos t - \cos 3t.$$

We will assume the particular solution has the form $Y(t) = A\cos t + B\cos 3t$. Differentiating the particular solution and substituting into the nonhomogeneous equation we obtain

$$\begin{aligned} -A\cos t - 9B\cos 3t + 16(A\cos t + B\cos 3t) &= \cos t - \cos 3t \\ 15A\cos t + 7B\cos 3t &= \cos t - \cos 3t \end{aligned}$$

which implies $A = 1/15$ and $B = -1/7$. Consequently the general solution to the nonhomogeneous differential equation is

$$y(t) = y_c(t) + Y(t) = c_1\cos 4t + c_2 \sin 4t + \frac{1}{15}\cos t - \frac{1}{7}\cos 3t.$$

2. Find the solution to the initial value problem below.

$$\begin{aligned} 6y'' + 5y' - 6y &= t \\ y(0) &= 2 \\ y'(0) &= 1 \end{aligned}$$

First we will find the complementary solution $y_c(t)$ to the ODE:

$$6y'' + 5y' - 6y = 0.$$

The characteristic equation has the form

$$\begin{aligned} 6r^2 + 5r - 6 &= 0 \\ (3r-2)(2r+3) &= 0 \end{aligned}$$

which implies $r_1 = -3/2$ and $r_2 = 2/3$. Therefore

$$y_c(t) = c_1 e^{-3t/2} + c_2 e^{2t/3}.$$

Assuming the particular solution has the form $Y(t) = At + B$, upon differentiating the particular solution and substituting into the nonhomogeneous equation we obtain

$$\begin{aligned} 6(0) + 5A - 6(At + B) &= t \\ -6At + (5A - 6B) &= t. \end{aligned}$$

Thus $A = -1/6$ and $B = -5/36$. Consequently the general solution to the nonhomogeneous differential equation is

$$y(t) = y_c(t) + Y(t) = c_1 e^{-3t/2} + c_2 e^{2t/3} - \frac{1}{6}t - \frac{5}{36}.$$

Now we may use the initial conditions to evaluate c_1 and c_2.

$$\begin{aligned} y(0) &= 2 = c_1 + c_2 - \frac{5}{36} \\ y'(0) &= 1 = -\frac{3}{2}c_1 + \frac{2}{3}c_2 - \frac{1}{6} \end{aligned}$$

Multiplying the first equation by $3/2$ and adding to the second equation yields $c_2 = 105/52$. Substituting this into the first equation gives $c_1 = 14/117$. Thus the solution to the initial value problem is

$$y(t) = \frac{14}{117}e^{-3t/2} + \frac{105}{52}e^{2t/3} - \frac{1}{6}t - \frac{5}{36}.$$

3. Find the solution to the initial value problem below.

$$\begin{aligned} y'' - 4y' + 5y &= 3e^{-2t} + 2t^2 \\ y(0) &= 0 \\ y'(0) &= 3 \end{aligned}$$

First we will find the complementary solution $y_c(t)$ to the ODE:

$$y'' - 4y' + 5y = 0.$$

The characteristic equation has the form

$$\begin{aligned} r^2 - 4r + 5 &= 0 \\ r &= \frac{4 \pm \sqrt{16 - 4(1)(5)}}{2} \\ &= 2 \pm i. \end{aligned}$$

Therefore

$$y_c(t) = e^{2t}(c_1 \cos t + c_2 \sin t).$$

Assuming the particular solution has the form $Y(t) = Ae^{-2t} + Bt^2 + Ct + D$, upon differentiating the particular solution and substituting into the nonhomogeneous equation we obtain

$$\begin{aligned} (4Ae^{-2t} + 2B) - 4(-2Ae^{-2t} + 2Bt + C) + 5(Ae^{-2t} + Bt^2 + Ct + D) &= 3e^{-2t} + 2t^2 \\ (4A + 8A + 5A)e^{-2t} + (5B)t^2 + (-8B + 5C)t + (2B - 4C + 5D) &= 3e^{-2t} + 2t^2 \\ (17A)e^{-2t} + (5B)t^2 + (-8B + 5C)t + (2B - 4C + 5D) &= 3e^{-2t} + 2t^2. \end{aligned}$$

Thus $A = 3/17$ and $B = 2/5$. We also obtain the pair of equations

$$\begin{aligned} 0 &= -8B + 5C = -\frac{16}{5} + 5C \\ 0 &= 2B - 4C + 5D = \frac{4}{5} - 4C + 5D. \end{aligned}$$

The first equation implies $C = 16/25$. Substituting this into the second equation produces $D = 44/125$. Consequently the general solution to the nonhomogeneous differential equation is

$$y(t) = y_c(t) + Y(t) = e^{2t}(c_1 \cos t + c_2 \sin t) + \frac{3}{17}e^{-2t} + \frac{2}{5}t^2 + \frac{16}{25}t + \frac{44}{125}.$$

Now we may use the initial conditions to evaluate c_1 and c_2.

$$\begin{aligned} y(0) &= 0 = c_1 + \frac{3}{17} + \frac{44}{125} \\ y'(0) &= 3 = 2c_1 + c_2 + \frac{16}{25} - \frac{6}{17} \end{aligned}$$

Solving the first equation yields $c_1 = -1123/2125$. Substituting this into the second equation gives $c_2 = 8011/2125$. Thus the solution to the initial value problem is

$$y(t) = e^{2t}\left(-\frac{1123}{2125}\cos t + \frac{8011}{2125}\sin t\right) + \frac{3}{17}e^{-2t} + \frac{2}{5}t^2 + \frac{16}{25}t + \frac{44}{125}.$$