Millersville University **Name** **Answer Key**

Department of Mathematics

MATH 365, *Ordinary Differential Equations*, Homework 07

March 11, 2009

Please answer the following questions. Answers without justifying work will receive no credit. Partial credit will be given as appropriate, do not leave any problem blank. Each problem is worth 10 points. Your completed assignment is due at class time on Friday, March 12, 2009.

1. Find the general solutions to the following ordinary differential equations. Use variation of parameters to determine the particular solutions.

   (a) $y'' + y' - 2y = 2\ln t$

   The complementary solution is found by solving

   $$y'' + y' - 2y = 0$$

   whose characteristic equation is

   $$0 = r^2 + r - 2 = (r+2)(r-1) \quad \Longrightarrow r_1 = -2 \quad \text{and} \quad r_2 = 1.$$

   Thus $y_c(t) = c_1 e^{-2t} + c_2 e^t$. The Wronskian of the solutions $y_1(t) = e^{-2t}$ and $y_2(t) = e^t$ is

   $$W(y_1, y_2)(t) = \begin{vmatrix} e^{-2t} & e^t \\ -2e^{-2t} & e^t \end{vmatrix} = e^{-2t}e^t - (-2e^{-2t})e^t = 3e^{-t}.$$

   Applying the variation of parameters formula produces

   $$\begin{aligned} \mu_1'(t) &= -\frac{(2\ln t)e^t}{3e^{-t}} \\ &= -\frac{2}{3}(\ln t)e^{2t} \\ \mu_1(t) &= -\frac{2}{3}\int_1^t (\ln s)e^{2s}\,ds \\ \mu_2'(t) &= \frac{(2\ln t)e^{-2t}}{3e^{-t}} \\ &= \frac{2}{3}(\ln t)e^{-t} \\ \mu_2(t) &= \frac{2}{3}\int_1^t (\ln s)e^{-s}\,ds. \end{aligned}$$

   Therefore the general solution to the nonhomogeneous ODE is

   $$y(t) = c_1 e^{-2t} + c_2 e^t - \frac{2e^{-2t}}{3}\int_1^t (\ln s)e^{2s}\,ds + \frac{2e^t}{3}\int_1^t (\ln s)e^{-s}\,ds.$$

(b) $y'' - 4y' + 4y = \frac{1}{2}\sqrt{t}$

The complementary solution is found by solving

$$y'' - 4y' + 4y = 0$$

whose characteristic equation is

$$0 = r^2 - 4r + 4 = (r-2)^2 \quad \Longrightarrow r = 2.$$

Thus $y_c(t) = c_1 e^{2t} + c_2 t e^{2t}$. The Wronskian of the solutions $y_1(t) = e^{2t}$ and $y_2(t) = te^{2t}$ is

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & (2t+1)e^{2t} \end{vmatrix} = e^{2t}(2t+1)e^{2t} - 2te^{2t}e^{2t} = e^{4t}.$$

Applying the variation of parameters formula produces

$$\begin{aligned}
\mu_1'(t) &= -\frac{\sqrt{t}te^{2t}}{2e^{4t}} \\
&= -\frac{1}{2}t^{3/2}e^{-2t} \\
\mu_1(t) &= -\frac{1}{2}\int_0^t s^{3/2}e^{-2s}\,ds \\
\mu_2'(t) &= \frac{\sqrt{t}e^{2t}}{2e^{4t}} \\
&= \frac{1}{2}\sqrt{t}e^{-2t} \\
\mu_2(t) &= \frac{1}{2}\int_0^t \sqrt{s}e^{-2s}\,ds.
\end{aligned}$$

Therefore the general solution to the nonhomogeneous ODE is

$$y(t) = c_1e^{2t} + c_2te^{2t} - \frac{e^{2t}}{2}\int_0^t s^{3/2}e^{-2s}\,ds + \frac{te^{2t}}{2}\int_0^t \sqrt{s}e^{-2s}\,ds.$$

(c) $y'' - y = e^{-t^2}$

The complementary solution is found by solving

$$y'' - y = 0$$

whose characteristic equation is

$$0 = r^2 - 1 = (r+1)(r-1) \quad \Longrightarrow r_1 = -1 \quad \text{and} \quad r_2 = 1.$$

Thus $y_c(t) = c_1 e^{-t} + c_2 e^t$. The Wronskian of the solutions $y_1(t) = e^{-t}$ and $y_2(t) = e^t$ is

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-t} & e^t \\ -e^{-t} & e^t \end{vmatrix} = e^{-t}e^t - (-e^{-t})e^t = 2.$$

Applying the variation of parameters formula produces

$$\begin{aligned}
\mu_1'(t) &= -\frac{e^{-t^2}e^t}{2} \\
&= -\frac{1}{2}e^{t-t^2} \\
\mu_1(t) &= -\frac{1}{2}\int_0^t e^{s-s^2}\,ds \\
\mu_2'(t) &= \frac{e^{-t^2}e^{-t}}{2} \\
&= \frac{1}{2}e^{-(t+t^2)} \\
\mu_2(t) &= \frac{1}{2}\int_0^t e^{-(s+s^2)}\,ds.
\end{aligned}$$

Therefore the general solution to the nonhomogeneous ODE is

$$y(t) = c_1 e^t + c_2 e^{-t} - \frac{e^{-t}}{2}\int_0^t e^{s-s^2}\,ds + \frac{e^t}{2}\int_0^t e^{-(s+s^2)}\,ds.$$

2. Find the solution to the initial value problem below.

$$\begin{aligned} y'' + a^2 y &= F(t) \\ y(0) &= 0 \\ y'(0) &= 0 \end{aligned}$$

Assume that $a > 0$ is a constant and that $F(t)$ is a continuous function.

The complementary solution is found by solving

$$y'' + a^2 y = 0$$

whose characteristic equation is

$$0 = r^2 + a^2 \quad \Longrightarrow r = 0 \pm ai,$$

where $i = \sqrt{-1}$. Thus $y_c(t) = c_1 \cos at + c_2 \sin at$. The Wronskian of the solutions $y_1(t) = cosat$ and $y_2(t) = \sin at$ is

$$W(y_1, y_2)(t) = \begin{vmatrix} \cos at & \sin at \\ -a \sin at & a \cos at \end{vmatrix} = a \cos^2 at - (-a \sin at) \cos at) = a.$$

Applying the variation of parameters formula produces

$$\begin{aligned} \mu_1'(t) &= -\frac{F(t) \sin at}{a} \\ &= -\frac{1}{a} F(t) \sin at \\ \mu_1(t) &= -\frac{1}{a} \int_0^t F(s) \sin as\, ds \\ \mu_2'(t) &= \frac{F(t) \cos at}{a} \\ &= \frac{1}{a} F(t) \cos at \\ \mu_2(t) &= \frac{1}{a} \int_0^t F(s) \cos as\, ds. \end{aligned}$$

Therefore the general solution to the nonhomogeneous ODE is

$$y(t) = c_1 \cos at + c_2 \sin at - \frac{1}{a} \cos at \int_0^t F(s) \sin as\, ds + \frac{1}{a} \sin at \int_0^t F(s) \cos as\, ds.$$

Using the initial condition

$$\begin{aligned} 0 &= y(0) \\ &= c_1 \end{aligned}$$

the solution simplifies to

$$y(t) = c_2 \sin at - \frac{1}{a} \cos at \int_0^t F(s) \sin as\, ds + \frac{1}{a} \sin at \int_0^t F(s) \cos as\, ds.$$

Its derivative is

$$\begin{aligned} y'(t) &= ac_2 \cos at + \sin at \int_0^t F(s) \sin as\, ds - \frac{1}{a} F(t) \cos at\ \sin at \\ &\quad + \cos at \int_0^t F(s) \cos as\, ds + \frac{1}{a} F(t) \sin at\ \cos at \\ &= ac_2 \cos at + \sin at \int_0^t F(s) \sin as\, ds + \cos at \int_0^t F(s) \cos as\, ds. \end{aligned}$$

Thus $y'(0) = 0 = ac_2$ implies $c_2 = 0$. Therefore the solution becomes

$$\begin{aligned} y(t) &= -\frac{1}{a} \cos at \int_0^t F(s) \sin as\, ds + \frac{1}{a} \sin at \int_0^t F(s) \cos as\, ds \\ &= \frac{1}{a} \int_0^t F(s) \sin at \cos as\, ds - \frac{1}{a} \int_0^t F(s) \cos at \sin as\, ds \\ &= \frac{1}{a} \int_0^t F(s) \left(\sin at \cos as - \cos at \sin as \right)\, ds \\ &= \frac{1}{a} \int_0^t F(s) \sin a(t-s)\, ds. \end{aligned}$$

3. Consider the nonhomogeneous second order linear ODE below.

$$t^2y'' - 2ty' + 2y = te^{-t}$$

Show that $y_c(t) = c_1t^2 + c_2t$ is a complementary solution to the equation. Use variation of parameters to find the general solution.

Substituting the complementary solution into the homogeneous version of the ODE yields

$$\begin{aligned}
t^2(c_1t^2 + c_2t)'' - 2t(c_1t^2 + c_2t)' + 2(c_1t^2 + c_2t) &= t^2(2c_1) - 2t(2c_1t + c_2) + 2(c_1t^2 + c_2t) \\
&= (2c_1 - 4c_1 + 2c_1)t^2 + (-2c_2 + 2c_2)t \\
&= 0
\end{aligned}$$

which implies $y_c(t)$ solves the homogeneous version of the ODE.

Putting the nonhomogeneous equation in standard form produces

$$y'' - \frac{2}{t}y' + \frac{2}{t^2}y = \frac{e^{-t}}{t}.$$

The Wronskian of the solutions $y_1(t) = t^2$ and $y_2(t) = t$ is

$$W(y_1, y_2)(t) = \begin{vmatrix} t^2 & t \\ 2t & 1 \end{vmatrix} = t^2 - 2t^2 = -t^2.$$

Applying the variation of parameters formula produces

$$\begin{aligned}
\mu_1'(t) &= -\frac{\frac{e^{-t}}{t}t}{-t^2} \\
&= \frac{1}{t^2e^t} \\
\mu_1(t) &= \int_1^t \frac{1}{s^2e^s}\,ds \\
\mu_2'(t) &= \frac{\frac{e^{-t}}{t}t^2}{-t^2} \\
&= -\frac{1}{te^t} \\
\mu_2(t) &= -\int_1^t \frac{1}{se^s}\,ds.
\end{aligned}$$

Therefore the general solution to the nonhomogeneous ODE is

$$y(t) = c_1t^2 + c_2t + t^2\int_1^t \frac{1}{s^2e^s}\,ds - t\int_1^t \frac{1}{se^s}\,ds.$$