

Please answer the following questions. Answers without justifying work will receive no credit. Partial credit will be given as appropriate, do not leave any problem blank. Each problem is worth 10 points. Your completed assignment is due at class time on Wednesday, April 1, 2009.

1. Solve the following differential equation by means of a power series solution centered at  $t_0 = 0$ . Clearly state the recurrence relation, find at least the first four terms in each of two linearly independent solutions, and if possible find the general term in each solution.

$$y'' + ty' + 2y = 0$$

Assuming  $y(t) = \sum_{n=0}^{\infty} a_n t^n$ , differentiating, and substituting into the ODE, we obtain

$$\begin{aligned} 0 &= \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + t \sum_{n=1}^{\infty} n a_n t^{n-1} + 2 \sum_{n=0}^{\infty} a_n t^n \\ &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n + \sum_{n=0}^{\infty} n a_n t^n + \sum_{n=0}^{\infty} 2a_n t^n \\ &= \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + (n+2)a_n] t^n. \end{aligned}$$

The recurrence relation is thus

$$\begin{aligned} (n+2)(n+1)a_{n+2} + (n+2)a_n &= 0 \\ a_{n+2} &= -\frac{a_n}{n+1}. \end{aligned}$$

If we assume  $a_0$  is arbitrary then

$$\begin{aligned} a_2 &= -\frac{a_0}{1} \\ a_4 &= -\frac{a_2}{3} = \frac{a_0}{(1)(3)} \\ a_6 &= -\frac{a_4}{5} = -\frac{a_0}{(1)(3)(5)} \\ &\vdots \\ a_{2n} &= \frac{a_0(-2)^n n!}{(2n)!}. \end{aligned}$$

If we assume  $a_1$  is arbitrary then

$$\begin{aligned}a_3 &= -\frac{a_1}{2} \\a_5 &= -\frac{a_3}{4} = \frac{a_1}{2^2(1)(2)} \\a_7 &= -\frac{a_5}{6} = -\frac{a_1}{2^3(1)(2)(3)} \\&\vdots \\a_{2n+1} &= \frac{a_1(-1)^n}{2^n n!}.\end{aligned}$$

Thus the general solution to the ODE can be written as

$$y(t) = a_0 \sum_{n=0}^{\infty} \frac{(-2)^n n!}{(2n)!} t^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} t^{2n+1}.$$

2. Solve the following differential equation by means of a power series solution centered at  $t_0 = 1$ . Clearly state the recurrence relation, find at least the first four terms in each of two linearly independent solutions, and if possible find the general term in each solution.

$$t y'' + y' + t y = 0$$

Assuming  $y(t) = \sum_{n=0}^{\infty} a_n(t-1)^n$ , differentiating, and substituting into the ODE, we obtain

$$\begin{aligned}
0 &= (t-1+1) \sum_{n=2}^{\infty} n(n-1)a_n(t-1)^{n-2} + \sum_{n=1}^{\infty} n a_n(t-1)^{n-1} + (t-1+1) \sum_{n=0}^{\infty} a_n(t-1)^n \\
&= \sum_{n=2}^{\infty} n(n-1)a_n(t-1)^{n-2} + (t-1) \sum_{n=2}^{\infty} n(n-1)a_n(t-1)^{n-2} + \sum_{n=1}^{\infty} n a_n(t-1)^{n-1} \\
&\quad + (t-1) \sum_{n=0}^{\infty} a_n(t-1)^n + \sum_{n=0}^{\infty} a_n(t-1)^n \\
&= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(t-1)^n + \sum_{n=2}^{\infty} n(n-1)a_n(t-1)^{n-1} + \sum_{n=1}^{\infty} n a_n(t-1)^{n-1} \\
&\quad + \sum_{n=0}^{\infty} a_n(t-1)^{n+1} + \sum_{n=0}^{\infty} a_n(t-1)^n \\
&= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(t-1)^n + \sum_{n=1}^{\infty} (n+1)n a_{n+1}(t-1)^n + \sum_{n=0}^{\infty} (n+1)a_{n+1}(t-1)^n \\
&\quad + \sum_{n=0}^{\infty} a_n(t-1)^{n+1} + \sum_{n=0}^{\infty} a_n(t-1)^n \\
&= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(t-1)^n + \sum_{n=0}^{\infty} (n+1)n a_{n+1}(t-1)^n + \sum_{n=0}^{\infty} (n+1)a_{n+1}(t-1)^n \\
&\quad + \sum_{n=1}^{\infty} a_{n-1}(t-1)^n + \sum_{n=0}^{\infty} a_n(t-1)^n \\
&= 2a_2 + a_1 + a_0 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2}(t-1)^n + \sum_{n=1}^{\infty} (n+1)n a_{n+1}(t-1)^n \\
&\quad + \sum_{n=1}^{\infty} (n+1)a_{n+1}(t-1)^n + \sum_{n=1}^{\infty} a_{n-1}(t-1)^n + \sum_{n=1}^{\infty} a_n(t-1)^n \\
&= 2a_2 + a_1 + a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} + (n+1)n a_{n+1} + (n+1)a_{n+1} + a_{n-1} + a_n] (t-1)^n \\
&= 2a_2 + a_1 + a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} + (n+1)^2 a_{n+1} + a_{n-1} + a_n] (t-1)^n
\end{aligned}$$

Therefore

$$a_2 = -\frac{1}{2}(a_0 + a_1)$$

and the recurrence relation for  $n \geq 1$  is

$$a_{n+2} = -\frac{(n+1)^2 a_{n+1} + a_n + a_{n-1}}{(n+2)(n+1)}.$$

If we assume  $a_0 = 1$  and  $a_1 = 0$  then we obtain the coefficients

$$\begin{aligned} a_2 &= -\frac{1}{2} \\ a_3 &= \frac{1}{6} \\ a_4 &= -\frac{1}{12}. \end{aligned}$$

Thus one solution to the ODE is

$$y_1(t) = 1 - \frac{1}{2}(t-1)^2 + \frac{1}{6}(t-1)^3 - \frac{1}{12}(t-1)^4 + \dots$$

If we assume  $a_0 = 0$  and  $a_1 = 1$  then we obtain the coefficients

$$\begin{aligned} a_2 &= -\frac{1}{2} \\ a_3 &= \frac{1}{6} \\ a_4 &= -\frac{1}{6} \end{aligned}$$

Thus another solution to the ODE is

$$y_2(t) = (t-1) - \frac{1}{2}(t-1)^2 + \frac{1}{6}(t-1)^3 - \frac{1}{6}(t-1)^4 + \dots$$

3. Solve the following differential equation by means of a power series solution centered at  $t_0 = 0$ . Clearly state the recurrence relation, find at least the first four terms in each of two linearly independent solutions, and if possible find the general term in each solution.

$$(1 + t^2)y'' - 2ty' + 3y = 0$$

Assuming  $y(t) = \sum_{n=0}^{\infty} a_n t^n$ , differentiating, and substituting into the ODE, we obtain

$$\begin{aligned} 0 &= (1 + t^2) \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - 2t \sum_{n=1}^{\infty} n a_n t^{n-1} + 3 \sum_{n=0}^{\infty} a_n t^n \\ &= \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n t^n - \sum_{n=1}^{\infty} 2n a_n t^n + \sum_{n=0}^{\infty} 3a_n t^n \\ &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n + \sum_{n=0}^{\infty} n(n-1)a_n t^n - \sum_{n=0}^{\infty} 2n a_n t^n + \sum_{n=0}^{\infty} 3a_n t^n \\ &= \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + (n^2 - 3n + 3)a_n] t^n. \end{aligned}$$

The recurrence relation is thus

$$\begin{aligned} (n+2)(n+1)a_{n+2} + (n^2 - 3n + 3)a_n &= 0 \\ a_{n+2} &= -\frac{n^2 - 3n + 3}{(n+2)(n+1)} a_n. \end{aligned}$$

Assuming  $a_0$  is arbitrary we obtain the coefficients

$$\begin{aligned} a_2 &= -\frac{3}{(1)(2)} a_0 = -\frac{3}{2} a_0 \\ a_4 &= -\frac{1}{(3)(4)} a_2 = \frac{1}{8} a_0 \\ a_6 &= -\frac{7}{(5)(6)} a_4 = -\frac{7}{240} a_0. \end{aligned}$$

Assuming  $a_1$  is arbitrary we obtain the coefficients

$$\begin{aligned} a_3 &= -\frac{1}{(2)(3)} a_1 = -\frac{1}{6} a_1 \\ a_5 &= -\frac{3}{(4)(5)} a_3 = \frac{1}{40} a_1 \\ a_7 &= -\frac{13}{(6)(7)} a_5 = -\frac{13}{1680} a_1. \end{aligned}$$

Thus the first four nonzero terms in each of two linearly independent solutions to the ODE resemble

$$y(t) = a_0 \left( 1 - \frac{3}{2}t^2 + \frac{1}{8}t^4 - \frac{7}{240}t^6 + \dots \right) + a_1 \left( t - \frac{1}{6}t^3 + \frac{1}{40}t^5 - \frac{13}{1680}t^7 + \dots \right).$$

4. Solve the following differential equation by means of a power series solution centered at  $t_0 = 0$ . Clearly state the recurrence relation, find at least the first four terms in each of two linearly independent solutions, and if possible find the general term in each solution.

$$(1 - t)y'' + ty' - y = 0$$

Assuming  $y(t) = \sum_{n=0}^{\infty} a_n t^n$ , differentiating, and substituting into the ODE, we obtain

$$\begin{aligned} 0 &= (1 - t) \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + t \sum_{n=1}^{\infty} n a_n t^{n-1} - \sum_{n=0}^{\infty} a_n t^n \\ &= \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n t^{n-1} + \sum_{n=1}^{\infty} n a_n t^n - \sum_{n=0}^{\infty} a_n t^n \\ &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n - \sum_{n=1}^{\infty} n(n+1)a_{n+1} t^n + \sum_{n=0}^{\infty} n a_n t^n - \sum_{n=0}^{\infty} a_n t^n \\ &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n - \sum_{n=0}^{\infty} n(n+1)a_{n+1} t^n + \sum_{n=0}^{\infty} n a_n t^n - \sum_{n=0}^{\infty} a_n t^n \\ &= \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - n(n+1)a_{n+1} + (n-1)a_n] t^n. \end{aligned}$$

The recurrence relation is then

$$\begin{aligned} (n+2)(n+1)a_{n+2} - n(n+1)a_{n+1} + (n-1)a_n &= 0 \\ a_{n+2} &= \frac{n(n+1)a_{n+1} - (n-1)a_n}{(n+2)(n+1)}. \end{aligned}$$

We can make determining the coefficients more convenient if we assume  $a_0 = 1$  and  $a_1 = 0$ . In this case

$$\begin{aligned} a_2 &= \frac{1}{(1)(2)} = \frac{1}{2!} \\ a_3 &= \frac{(1)(2)a_2}{(2)(3)} = \frac{1}{3!} \\ a_4 &= \frac{(2)(3)a_3 - a_2}{(3)(4)} = \frac{1}{4!} \\ &\vdots \\ a_n &= \frac{1}{n!}. \end{aligned}$$

Thus one solution to the ODE is

$$y_1(t) = 1 + \sum_{n=2}^{\infty} \frac{t^n}{n!} = e^t - t.$$

Now if we let  $a_0 = 0$  and  $a_1 = 1$  we get

$$\begin{aligned}a_2 &= 0 \\a_3 &= 0 \\&\vdots \\a_n &= 0.\end{aligned}$$

Thus a second solution to the ODE is

$$y_2(t) = t.$$