Millersville University **Name** **Answer Key**
Department of Mathematics
MATH 365, *Ordinary Differential Equations*, Homework 09
April 08, 2009

Please answer the following questions. Answers without justifying work will receive no credit. Partial credit will be given as appropriate, do not leave any problem blank. Each problem is worth 10 points. Your completed assignment is due at class time on Friday, April 10, 2009.

1. Determine the regular and irregular singular points (if any) of the following second order linear ordinary differential equation.

$$t^3(1-t)y'' + (3t+2)y' + t^4 y = 0$$

The singular points are the solutions of

$$t^3(1-t) = 0,$$

namely $t = 0$ and $t = 1$.

Consider $t = 0$.

$$\begin{aligned} \lim_{t\to 0} t\left(\frac{3t+2}{t^3(1-t)}\right) &= \lim_{t\to 0} \frac{3t+2}{t^2(1-t)} \quad \text{does not exist, and} \\ \lim_{t\to 0} t^2\left(\frac{t^4}{t^3(1-t)}\right) &= \lim_{t\to 0} t^3 = 0 \end{aligned}$$

Thus $t = 0$ is an irregular singular point.

Now consider $t = 1$.

$$\begin{aligned} \lim_{t\to 1}(t-1)\left(\frac{3t+2}{t^3(1-t)}\right) &= \lim_{t\to 1} -\frac{3t+2}{t^2} = -5 \\ \lim_{t\to 1}(t-1)^2\left(\frac{t^4}{t^3(1-t)}\right) &= \lim_{t\to 1} t(1-t) = 0 \end{aligned}$$

Thus $t = 1$ is a regular singular point.

2. Find the general solution of the following ordinary differential equation.

$$t^2y'' + ty' - 9y = 0$$

If $t = e^z$ then

$$\frac{d^2y}{dz^2} - 9y = 0$$

whose characteristic equation is

$$r^2 - 9 = 0$$

which implies the characteristic roots are $r_1 = -3$ and $r_2 = 3$. Thus the general solution to the ODE is

$$y(t) = c_1t^{-3} + c_2t^3.$$

3. Find the general solution of the following ordinary differential equation.

$$t^2y'' + 3t\,y' - 3y = t^2 - 4t + 2$$

The complementary solution can be found by solving the homogeneous ODE:

$$t^2y'' + 3t\,y' - 3y = 0.$$

If $t = e^z$ then

$$\frac{d^2y}{dz^2} + 2\frac{dy}{dz} - 3y = 0$$

whose characteristic equation is

$$0 = r^2 + 2r - 3 = (r + 3)(r - 1)$$

which implies the characteristic roots are $r_1 = -3$ and $r_2 = 1$. Thus the complementary solution to the ODE is

$$y_c(t) = c_1t^{-3} + c_2t.$$

The Wronskian of these two solutions is

$$W(t) = \begin{vmatrix} t^{-3} & t \\ -3t^{-4} & 1 \end{vmatrix} = t^{-3} + 3t^{-3} = 4t^{-3}.$$

Applying the technique of variation of parameters

$$\begin{aligned}
\mu_1'(t) &= -\frac{\left(\frac{t^2-4t+2}{t^2}\right)t}{4t^{-3}} = -\frac{1}{4}t^4 + t^3 - \frac{1}{2}t^2 \\
\mu_1(t) &= -\frac{1}{20}t^5 + \frac{1}{4}t^4 - \frac{1}{6}t^3 \\
\mu_2'(t) &= \frac{\left(\frac{t^2-4t+2}{t^2}\right)t^{-3}}{4t^{-3}} = \frac{1}{4} - \frac{1}{t} + \frac{1}{2t^2} \\
\mu_2(t) &= \frac{1}{4}t - \ln t - \frac{1}{2t}
\end{aligned}$$

Thus the particular solution has the form

$$\begin{aligned}
Y(t) &= \left(-\frac{1}{20}t^5 + \frac{1}{4}t^4 - \frac{1}{6}t^3\right)t^{-3} + \left(\frac{1}{4}t - \ln t - \frac{1}{2t}\right)t \\
&= -\frac{1}{20}t^2 + \frac{1}{4}t - \frac{1}{6} + \frac{1}{4}t^2 - t\ln t - \frac{1}{2} \\
&= \frac{1}{5}t^2 + \frac{1}{4}t - t\ln t - \frac{2}{3}.
\end{aligned}$$

Consequently the general solution to the ODE is

$$y(t) = c_1t^{-3} + c_2t + \frac{1}{5}t^2 - t\ln t - \frac{2}{3}.$$

4. Find a series solution near $t_0 = 0$ to the following ordinary differential equation.

$$t^2 y'' + t\, y' + (t^2 - 1)y = 0$$

Note that $t_0 = 0$ is a singular point and that

$$\begin{aligned} \lim_{t \to 0} t \left(\frac{t}{t^2} \right) &= \lim_{t \to 0} 1 = 1 \\ \lim_{t \to 0} t^2 \left(\frac{t^2 - 1}{t^2} \right) &= \lim_{t \to 0} (t^2 - 1) = -1 \end{aligned}$$

so $t_0 = 0$ is a regular singular point. The exponents of singularity are the solutions of the indicial equation

$$0 = r(r-1) + (1)r + (-1) = r^2 - 1.$$

Thus $r = \pm 1$. We can always find a series solution corresponding to the larger of the exponents of singularity, consequently we will seek a solution of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^{n+1}.$$

Differentiating this solution and substituting it into the ODE produces:

$$\begin{aligned} 0 &= t^2 \sum_{n=0}^{\infty} (n+1) n a_n t^{n-1} + t \sum_{n=0}^{\infty} (n+1) a_n t^n + (t^2 - 1) \sum_{n=0}^{\infty} a_n t^{n+1} \\ &= \sum_{n=0}^{\infty} (n+1) n a_n t^{n+1} + \sum_{n=0}^{\infty} (n+1) a_n t^{n+1} - \sum_{n=0}^{\infty} a_n t^{n+1} + \sum_{n=0}^{\infty} a_n t^{n+3} \\ &= \sum_{n=0}^{\infty} \left[(n+1)n + (n+1) - 1 \right] a_n t^{n+1} + \sum_{n=2}^{\infty} a_{n-2} t^{n+1} \\ &= \sum_{n=0}^{\infty} n(n+2) a_n t^{n+1} + \sum_{n=2}^{\infty} a_{n-2} t^{n+1} \\ &= 3a_1 t^2 + \sum_{n=2}^{\infty} n(n+2) a_n t^{n+1} + \sum_{n=2}^{\infty} a_{n-2} t^{n+1} \\ &= 3a_1 t^2 + \sum_{n=2}^{\infty} \left[n(n+2) a_n + a_{n-2} \right] t^{n+1} \end{aligned}$$

Thus and $a_1 = 0$ and the recurrence relation is

$$a_n = -\frac{a_{n-2}}{n(n+2)} \quad \text{for } n \geq 2.$$

If we let $a_0 = 1$ then we see that

$$\begin{aligned} a_2 &= -\frac{1}{2(4)} \\ a_4 &= -\frac{a_2}{4(6)} = \frac{1}{2(4^2)6} \\ a_6 &= -\frac{a_4}{6(8)} = -\frac{1}{2(4^2)(6^2)8} \\ &\vdots \\ a_{2n} &= \frac{(-1)^n}{2(4^2)(6^2)\cdots(2n)^2(2n+2)}. \end{aligned}$$

Since $a_1 = 0$ then $a_{2n+1} = 0$ for $n \geq 1$. Thus a solution to the ODE is

$$y(t) = t - \frac{t^3}{2(4)} + \frac{t^5}{2(4^2)6} + \cdots + \frac{(-1)^n t^{2n+1}}{2(4^2)(6^2)\cdots(2n)^2(2n+2)} + \cdots.$$