First Order Linear Equations and Integrating Factors

MATH 365, Ordinary Differential Equations

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Introduction

A first order linear ordinary differential equation has the general form:

\[
\frac{dy}{dt} + p(t) y = g(t)
\]

If \( p(t) \) and \( g(t) \) are constants, as in

\[
\frac{dy}{dt} + a y = b
\]

we say the first order linear equation is a constant coefficient ordinary differential equation.

Integrating Factor

Given the first order linear ordinary differential equation

\[
\frac{dy}{dt} + p(t) y = g(t)
\]

an integrating factor is a function of the form

\[
\mu(t) = e^{\int p(s) \, ds}
\]

If we multiply both sides of the ODE by \( \mu(t) \), the left-hand side of the ODE becomes an exact derivative.

\[
\mu(t) \left( \frac{dy}{dt} + p(t) y \right) = \mu(t) g(t)
\]

\[
e^{\int p(s) \, ds} \left( \frac{dy}{dt} + p(t) y \right) = e^{\int p(s) \, ds} g(t)
\]

\[
e^{\int p(s) \, ds} \frac{dy}{dt} + p(t) e^{\int p(s) \, ds} y = e^{\int p(s) \, ds} g(t)
\]

\[
\frac{d}{dt} \left( e^{\int p(s) \, ds} y \right) = e^{\int p(s) \, ds} g(t)
\]
Example

Find the general solution of the first order linear ordinary differential equation:

\[
\frac{dy}{dt} = e^{-2t} + y
\]

In[1]:= soln = Table[DSolve[{y'[t] == Exp[-2 t] + y[t], y[0] == c}, y[t], t, {c, {-3, -2, -1, 0, 1, 2, 3}}];

In[2]:= Plot[y[t] /. soln, {t, 0, 5}, AxesLabel -> {"t", "y(t)"}]

Example

Find the general solution of the first order linear ordinary differential equation:

\[
\frac{dy}{dt} = 3t - y
\]

In[3]:= soln = Table[DSolve[{y'[t] == 3 t - y[t], y[0] == c}, y[t], t, {c, {-3, -2, -1, 0, 1, 2, 3}}];

In[4]:= Plot[y[t] /. soln, {t, 0, 1}, AxesLabel -> {"t", "y(t)"}]
Example

Find the general solution of the first order linear ordinary differential equation:

\[
\frac{dy}{dt} - \frac{y}{t} = \cos t
\]

In[5]: = soln = Table[DSolve[{y'[t] - y[t]/t == Cos[t], y[\[Pi]/2] == c}, y[t], t], {c, {-3, -2, -1, 0, 1, 2, 3}}];

In[6]: = Plot[y[t] /. soln, {t, \[Pi]/4, 4 \[Pi]}, AxesLabel -> {"t", "y(t)"}]

Homework

Read Section 2.1

Pages 39-41: work exercises 1c-11c odd, 13-19 odd, 31, 33