Radioactive Decay

Radioactive decay takes place continuously. The number of atoms decaying at any instant is proportional to the number of un-decayed atoms present. Let $Q$ represent the number of un-decayed atoms and $t$ represent time. A first-order ODE describing radioactive decay is

$$\frac{dQ}{dt} = -kQ$$

The proportionality constant $k$ is called the decay constant.

If $Q(t_0) = Q_0$ then the solution to the initial value problem is $Q(t) = Q_0 e^{-k(t-t_0)}$.

The half-life of the radioactive substance is the time required for half of the initial amount to decay. Find the relationship between the half-life and the decay constant.

Continuous Infusion

Suppose that a chemical is eliminated from the blood stream of an animal at a rate proportional to the amount of the chemical present (similar to radioactive decay). The chemical enters the animal's blood stream at a rate given by the function $F(t)$. A first order differential equation describing the amount of chemical in the blood stream is then

$$\frac{dQ}{dt} = F(t) - kQ$$

- Example (1 of 2)

Suppose the chemical has a half-life of 2.5 hours in the blood stream and enters the blood stream at a constant rate $C$. If the chemical is a beneficial medicine whose effective dose is 50±0.5 mg, what should the infusion rate be?

In[1]:= soln = DSolve[$\{q'[t] = c - (\text{Log}[2] / 2.5) q[t], q[0] = 0, q[t], t \}$]

Out[1]= $\{q[t] \to c e^{-0.277259 t} (-3.60674 + 3.60674 c e^{0.277259 t})\}$

In[2]:= Limit[q[t] /. First[soln], t \to \text{Infinity}]

Out[2]= 3.60674 c
In[3]:= Solve[@50, c]
Out[3]= \{c \to 13.8629\}

How long after the infusion is started will it take until the effective dose is achieved?

In[4]:= Plot[@50 - 2.5, t /. First[soln] /. First[%], \{t, 0, 20\}]
Out[4]=

In[5]:= Solve[50 - 2.5 = q[t] / . First[soln] / . First[%], t]

Solve::ifun:
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. \[\]
Out[5]= \{\{t \to 10.8048\}\}

■ Example (2 of 2)

Suppose the chemical has a half-life of 3 hours in the blood stream and enters the blood stream at a rate given by 10(1 + sin (2\pi \ t)). Initially 50 mg of the chemical is in the blood stream. How much of the chemical is in the blood stream at time \(t > 0\)?

In[6]:= soln = D Solve[@q[t] = 10 (1 + Sin[2 \pi \ t]) - (Log[2] / 3) q[t], q[0] = 50], q[t], t]

Out[6]= \{\{q[t] \to
\left\{5 \times 2^{1 \frac{\pi}{3}} \left(-108 \pi^2 + 27 \times 2^{1 \frac{\pi}{3}} \pi^2 + 18 \pi \log[2] + 180 \pi^2 \log[2] - 9 \times 2^{1 \frac{\pi}{3}} \pi \cos[2 \pi \ t] \log[2] - 3 \log[2]^2 + 3 \times 2^{\frac{2}{3}} \log[2]^2 + 5 \log[2]^3 + 3 \times 2^{\frac{2}{3}} \log[2]^2 \sin[2 \pi \ t]\right)\right\} / \left(\log[2] \left(36 \pi^2 + \log[2]^2\right)\right)\}\}
Stirred/Mixed Tanks

Suppose a tank (or some other vessel) has a constant volume $V$ and is being fed a solution of water and a chemical in concentration $\gamma$ g/L at a rate of $r$ L/min. The tank is well stirred so that the chemical is evenly distributed throughout the tank. The well-stirred mixture flows out of the tank also at rate $r$ L/min. If the tank is initially filled completely with pure water, find the concentration of the chemical in the well-stirred tank as a function of time.

We know the concentration of the chemical in the incoming stream and we know the flow rate of the incoming stream. Therefore the amount of chemical entering the tank per minute is $(\gamma r)$ g/min.

We can express the concentration of the chemical in the tank as the amount of chemical in the tank divided by the volume of the tank. Thus the concentration can be expressed as $Q(t)/V$. Since the concentration of the chemical in the outflow is the same as the concentration of the chemical in the tank (recall that the tank is well-stirred), the amount of chemical leaving the tank per minute is $r Q(t)/V$ g/min.

Since the rate of change in the amount of the chemical is the difference between the inflow rate and the outflow rate, then an ODE describing the amount of chemical in the tank is

$$\frac{dQ}{dt} = \gamma r - r Q / V$$

Since the concentration $C(t)$, of the chemical in the tank is just the amount of chemical in the tank divided by the volume of the tank (in other words $C(t)=Q(t)/V$), an ODE for the concentration is

$$\frac{1}{V} \frac{dQ}{dt} = \frac{1}{V} \left( \gamma r - r Q / V \right)$$

$$\frac{dC}{dt} = \frac{\gamma r}{V} - \frac{r C}{V}$$

- Example

Suppose $r = 2$ L/min and $\gamma = 1/2$ g/L and $V = 500$ L. Find the concentration of chemical in the tank as a function of time.

$$\frac{dC}{dt} = \frac{1}{2} \frac{1}{c[t]} \left( 1 - \frac{2}{250} e^{t/250} \right)$$

In[8]:= \text{soln = DSolve}\{c[0] = 0, c'[t] = (1/2) (2) / 500 - (2 / 500) c[t]\}, c[t], t\}

Out[8]= \{\{c[t] \to \frac{1}{2} e^{-t/250} \left( -1 + e^{t/250} \right) \}\}
Newton's Law of Cooling

Suppose the temperature of an object is \( \theta \) and the temperature of the environment in which the object sits is \( T \). Furthermore assume the temperature of the environment is constant. Newton's Law of Cooling states that the rate of change in the temperature of the object is proportional to the difference between the temperature of the object and the temperature of the environment. Thus an ODE describing the change is \( \theta \) is

\[
\frac{d\theta}{dt} = -k (\theta - T)
\]

If we assume the proportionality constant \( k > 0 \), why does the right hand side possess the minus "-" sign?

**Example**

Suppose a cup of coffee is initially 200 degrees Fahrenheit and one minute later is at 190 degrees Fahrenheit in a room whose temperature is 70 degrees Fahrenheit. When will the temperature of the coffee be 150 degrees Fahrenheit?

\[
\text{In[10]} = \text{soln} = \text{DSolve}[\{\theta[0] = 200, \ \theta'[t] = -k (\theta[t] - 70)\}, \theta[t], t]
\]

\[
\text{Out[10]} = \{\{\theta[t] \rightarrow 10 e^{-k t} \left[13 + 7 e^{k t}\right]\}\}
\]

\[
\text{In[11]} = \text{Solve}[190 == \theta[t] / \text{. First[soln]} / . t \rightarrow 1, k]
\]

Solve::ifun :
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. 

\[
\text{Out[11]} = \{\{k \rightarrow \text{Log}\left[\frac{13}{12}\right]\}\}\}
\]

\[
\text{In[12]} = \text{N[\%]}
\]

\[
\text{Out[12]} = \{\{k \rightarrow 0.0800427\}\}\]
Escape Velocity

An object of mass $m$ is launched perpendicularly from the surface of the earth with initial velocity $v_0$. The only force on the object is gravity which varies inversely with the square of the distance separating the centers of mass of the object and the earth. Assume the radius of the earth in some system of units is $R$. Let the distance from the object to the surface of the earth be $x$. Then according to Newton's Second Law of Motion ($F = ma$)

$$m \frac{dv}{dt} = -\frac{mgR^2}{(R + x)^2}$$

This ODE contains two dependent variables ($v$ and $x$). We must re-write the ODE by treating $x$ as the independent variable.

By the Chain Rule for derivatives:

$$\frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = v \frac{dv}{dx}$$

Thus the ODE above can be written and solved as follows.
\[
\frac{dv}{dt} = -\frac{mg}{(R+x)^2}
\]
\[
\frac{dv}{dt} = -\frac{gR^2}{(R+x)^2}
\]
\[
\frac{dv}{dx} = -\frac{gR^2}{(R+x)^2}
\]
\[
v \frac{dv}{dx} = -\frac{gR^2}{(R+x)^2}
\]
\[
\int_v^u du = -\int_x^x \frac{gR^2}{(R+s)^2} ds
\]
\[
\frac{1}{2} (v^2 - v_0^2) = \frac{gR^2}{R+x} - \frac{gR^2}{R}
\]
\[
v (x) = \sqrt{v_0^2 - 2gR + \frac{2gR}{R+x}}
\]

In[16]:= \[v[x_, v0_] := \text{Sqrt}\left[v0^2 - 2gR + \frac{2gR^2}{R+x}\right]\]

Maximum altitude is reached when \(v=0\).

In[17]:= \text{alt} = \text{Solve}[v(x, v0) = 0, x]

Out[17]= \(
\text{\{\text{x} \to \frac{Rv0^2}{2gR - v0^2}\}}\)

In[18]:= \text{iv} = \text{Solve}[v(x, v0) = 0, v0]

Out[18]= \(
\text{\{\text{v0} \to -\sqrt{2\sqrt{gR - \frac{gR^2}{R+x}}}, \text{v0} \to \sqrt{2\sqrt{gR - \frac{gR^2}{R+x}}}\}}\)

In[19]:= \text{Needs["Units"]}; \text{Needs["PhysicalConstants"]}

In[20]:= \text{EarthRadius}

Out[20]= 6378140 \text{ Meter}

In[21]:= \text{AccelerationDueToGravity}

Out[21]= 196133 \text{ Meter}

Out[21]= 20000 \text{ Second}^2

In[22]:= \text{R = 6378140; g = 9.80665;}
In[23]:= \( \text{Plot}[x / . \text{First}[alt], \{v0, 0, 11000\}, \text{AxesLabel} \to \{"v0", "x_{\text{max}}"\}] \)

\[ x_{\text{max}} \]

\[ 2000 \quad 4000 \quad 6000 \quad 8000 \quad 10000 \]

\[ v0 \]

5.0 \times 10^6

1.0 \times 10^7

1.5 \times 10^7

2.0 \times 10^7

2.5 \times 10^7

3.0 \times 10^7

Out[23]=

In[24]:= \( \text{Limit}[v0 / . \text{iv}[2]], x \to \text{Infinity} \)

Out[24]= 11184.6

In[25]:= \( \text{Convert}[11184.6 \text{ Meter / Second}, \text{ Mile / Hour}] \)

Out[25]= 25019.2 \text{ Mile / Hour}

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**Homework**

Read Section 2.3

Pages 59-68: work exercises 1, 3, 15, 19, 31, 32