Series Solutions Near an Ordinary Point

MATH 365 Ordinary Differential Equations

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Consider the second order linear homogeneous ODE:

\[ P(t)y'' + Q(t)y' + R(t)y = 0 \]

where \( P, Q, \) and \( R \) are polynomials.

**Definition**
A point \( t_0 \) such that \( P(t_0) \neq 0 \) is called an ordinary point. If \( P(t_0) = 0 \) then \( t_0 \) is called a singular point.
Ordinary Points (2 of 2)

\[ P(t)y'' + Q(t)y' + R(t)y = 0 \]

If \( t_0 \) is an ordinary point, then by continuity there exists an interval \((a, b)\) containing \( t_0 \) on which \( P(t) \neq 0 \) for all \( t \in (a, b) \).

Thus the functions \( p(t) = \frac{Q(t)}{P(t)} \) and \( q(t) = \frac{R(t)}{P(t)} \) are defined and continuous on \((a, b)\) and the ODE can be written as

\[ y'' + p(t)y' + q(t)y = 0. \]

If the initial conditions are \( y(t_0) = y_0 \) and \( y'(t_0) = y'_0 \) then there exists a unique solution to the ODE satisfying the initial conditions.
Consider the ODE
\[ y'' - 4y = 0 \]
and find a power series solution with positive radius of convergence centered at an ordinary point.
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\[ y'' - 4y = 0 \]

and find a power series solution with positive radius of convergence centered at an ordinary point. Let \( t_0 = 0 \) be the ordinary point for simplicity and

\[
y(t) = \sum_{n=0}^{\infty} a_n t^n
\]
Consider the ODE

\[ y'' - 4y = 0 \]

and find a power series solution with positive radius of convergence centered at an ordinary point. Let \( t_0 = 0 \) be the ordinary point for simplicity and

\[
\begin{align*}
y(t) &= \sum_{n=0}^{\infty} a_n t^n \\
y'(t) &= \sum_{n=1}^{\infty} n a_n t^{n-1} \\
y''(t) &= \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}
\end{align*}
\]

Substitute into the ODE.
Power Series Solutions (2 of 5)

0 = y'' - 4y
0 = y'' − 4y

= ∑_{n=2}^{∞} n(n − 1)a_n t^{n−2} − 4 ∑_{n=0}^{∞} a_n t^n
0 = y'' - 4y

= \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - 4 \sum_{n=0}^{\infty} a_n t^n

= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n - 4 \sum_{n=0}^{\infty} a_n t^n

= \sum_{n=0}^{\infty} [a_{n+2}(n+2)(n+1) - 4a_n] t^n
Power Series Solutions (2 of 5)

\[ 0 = y'' - 4y \]
\[ = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - 4 \sum_{n=0}^{\infty} a_n t^n \]
\[ = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n - 4 \sum_{n=0}^{\infty} a_n t^n \]
\[ = \sum_{n=0}^{\infty} [a_{n+2}(n+2)(n+1) - 4a_n] t^n \]

\[ 0 = a_{n+2}(n+2)(n+1) - 4a_n \quad \text{(for } n = 0, 1, 2, \ldots) \]

\[ a_{n+2} = \frac{2^2 a_n}{(n+2)(n+1)} \]

This last equation is called a recurrence relation.
Let $a_0$ be arbitrary, then

\[
\begin{align*}
    a_2 & = \frac{2^2}{(2)(1)}a_0 = \frac{2^2}{2!}a_0 \\
    a_4 & = \frac{2^2}{(4)(3)}a_2 = \frac{2^4}{4!}a_0 \\
    a_6 & = \frac{2^2}{(6)(5)}a_4 = \frac{2^6}{6!}a_0 \\
    \vdots & \\
    a_{2n} & = \frac{2^{2n}}{(2n)!}a_0.
\end{align*}
\]
Let $a_1$ be arbitrary, then

\[
\begin{align*}
  a_3 &= \frac{2^2}{(3)(2)} a_1 = \frac{2^2}{3!} a_1 \\
  a_5 &= \frac{2^2}{(5)(4)} a_3 = \frac{2^4}{5!} a_1 \\
  a_7 &= \frac{2^2}{(7)(6)} a_5 = \frac{2^6}{7!} a_1 \\
  &\vdots \\
  a_{2n+1} &= \frac{2^{2n}}{(2n + 1)!} a_1.
\end{align*}
\]
Thus the general solution to \( y'' - 4y = 0 \) can be written as

\[
y(t) = \sum_{n=0}^{\infty} a_{2n} t^{2n} + \sum_{n=0}^{\infty} a_{2n+1} t^{2n+1}
\]

\[
= a_0 \sum_{n=0}^{\infty} \frac{2^{2n}}{(2n)!} t^{2n} + a_1 \sum_{n=0}^{\infty} \frac{2^{2n}}{(2n + 1)!} t^{2n+1}
\]

\[
= a_0 \sum_{n=0}^{\infty} \frac{(2t)^{2n}}{(2n)!} + \frac{a_1}{2} \sum_{n=0}^{\infty} \frac{(2t)^{2n+1}}{(2n + 1)!}
\]

\[
= a_0 \cosh(2t) + \frac{a_1}{2} \sinh(2t)
\]

We can confirm this series converges for all \( t \in \mathbb{R} \).
Airy’s Equation (1 of 6)

Find a power series solution about the ordinary point $t_0 = 0$ to Airy’s equation:

$$y'' - t y = 0.$$
Airy's Equation (1 of 6)

Find a power series solution about the ordinary point $t_0 = 0$ to Airy's equation:

$$y'' - t\,y = 0.$$

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

$$y'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$

$$y''(t) = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}.$$  

Substitute into the ODE.
Airy’s Equation (2 of 6)

\[ 0 = y'' - ty \]

\[ = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - t \sum_{n=0}^{\infty} a_n t^n \]

\[ = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n - \sum_{n=0}^{\infty} a_n t^{n+1} \]

\[ = 2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2} t^n - \sum_{n=1}^{\infty} a_{n-1} t^n \]

\[ = 2a_2 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} - a_{n-1}] t^n \]
Airy’s Equation (3 of 6)

Thus

\[
0 = 2a_2 \\
0 = (n + 2)(n + 1)a_{n+2} - a_{n-1} \quad \text{for } n = 1, 2, \ldots
\]

From the last equation we obtain the recurrence relation:

\[
a_{n+2} = \frac{a_{n-1}}{(n + 2)(n + 1)}.
\]
Thus

\[ 0 = 2a_2 \]
\[ 0 = (n + 2)(n + 1)a_{n+2} - a_{n-1} \quad (\text{for } n = 1, 2, \ldots) \]

From the last equation we obtain the recurrence relation:

\[ a_{n+2} = \frac{a_{n-1}}{(n + 2)(n + 1)}. \]

Since \( a_2 = 0 \) then \( a_5 = a_8 = a_{11} = \cdots = 0. \)
Let $a_0$ be arbitrary then

\[
\begin{align*}
    a_3 &= \frac{a_0}{2 \cdot 3} \\
    a_6 &= \frac{a_3}{5 \cdot 6} = \frac{a_0}{6 \cdot 5 \cdot 3 \cdot 2} \\
    a_9 &= \frac{a_6}{8 \cdot 9} = \frac{a_0}{9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2} \\
    &\vdots \\
    a_{3n} &= \frac{a_0}{(2)(3)(5)(6) \cdots (3n - 4)(3n - 3)(3n - 1)(3n)}.
\end{align*}
\]
Airy’s Equation (5 of 6)

Let $a_1$ be arbitrary then

\[ a_4 = \frac{a_1}{3 \cdot 4} \]
\[ a_7 = \frac{a_4}{6 \cdot 7} = \frac{a_1}{7 \cdot 6 \cdot 4 \cdot 3} \]
\[ a_{10} = \frac{a_7}{9 \cdot 10} = \frac{a_1}{10 \cdot 9 \cdot 7 \cdot 6 \cdot 4 \cdot 3} \]
\[ \vdots \]
\[ a_{3n+1} = \frac{a_1}{(3)(4)(6)(7) \cdots (3n - 3)(3n - 2)(3n)(3n + 1)} . \]
Thus the solution to Airy’s equation

\[ y'' - ty = 0 \]

is

\[
y(t) = a_0 \left[ 1 + \sum_{n=1}^{\infty} \frac{t^{3n}}{(2)(3)(5)(6) \cdots (3n-4)(3n-3)(3n-1)(3n)} \right]
\]

\[
+ a_1 \left[ t + \sum_{n=1}^{\infty} \frac{t^{3n+1}}{(3)(4)(6)(7) \cdots (3n-3)(3n-2)(3n)(3n+1)} \right]
\]

\[ = a_0 y_1(t) + a_1 y_2(t). \]
Homework

- Read Section 5.2
- Exercises: 1, 2, 3, 5, 6, 21
We have determined a method for solving an ODE of the form

\[ P(t)y'' + Q(t)y' + R(t)y = 0 \]

where \( P, Q, \) and \( R \) are polynomials.

The solution is a power series of the form \( y(t) = \sum_{n=0}^{\infty} a_n(t - t_0)^n \)

where \( t_0 \) is an ordinary point.
We have determined a method for solving an ODE of the form

\[ P(t) y'' + Q(t) y' + R(t) y = 0 \]

where \( P, Q, \) and \( R \) are polynomials.

The solution is a power series of the form

\[ y(t) = \sum_{n=0}^{\infty} a_n (t - t_0)^n \]

where \( t_0 \) is an ordinary point.

Today we extend this work to a broader range of functions than polynomial \( P, Q, \) and \( R. \)
Differentiation of Power Series

Suppose \( \phi(t) = \sum_{n=0}^{\infty} a_n(t - t_0)^n \) has a positive radius of convergence, then we may differentiate the series term-by-term.

\[
\begin{array}{c|c|c}
 n & \phi^{(n)}(t) & \phi^{(n)}(t_0) \\
\hline
 0 & \sum_{n=0}^{\infty} a_n(t - t_0)^n & a_0 \\
 1 & \sum_{n=1}^{\infty} n a_n(t - t_0)^{n-1} & a_1 \\
 2 & \sum_{n=2}^{\infty} n(n - 1) a_n(t - t_0)^{n-2} & 2a_2 \\
 \vdots & \vdots & \vdots \\
 m & \sum_{n=m}^{\infty} n(n - 1) \cdots (n - m + 1) a_n(t - t_0)^{n-m} & m! a_m \\
\end{array}
\]
Solution to an ODE

Suppose $\phi(t)$ solves the ODE $y'' + p(t)y' + q(t)y = 0$ then

$$0 = \phi''(t) + p(t)\phi'(t) + q(t)\phi(t)$$

$$\phi''(t) = -p(t)\phi'(t) - q(t)\phi(t)$$
Suppose $\phi(t)$ solves the ODE $y'' + p(t)y' + q(t)y = 0$ then

\[
0 = \phi''(t) + p(t)\phi'(t) + q(t)\phi(t) \\
\phi''(t) = -p(t)\phi'(t) - q(t)\phi(t) \\
\phi''(t_0) = -p(t_0)\phi'(t_0) - q(t_0)\phi(t_0)
\]
Solution to an ODE

Suppose \( \phi(t) \) solves the ODE \( y'' + p(t)y' + q(t)y = 0 \) then

\[
0 = \phi''(t) + p(t)\phi'(t) + q(t)\phi(t)
\]

\[
\phi''(t) = -p(t)\phi'(t) - q(t)\phi(t)
\]

\[
\phi''(t_0) = -p(t_0)\phi'(t_0) - q(t_0)\phi(t_0)
\]

\[
2!a_2 = -p(t_0)a_1 - q(t_0)a_0.
\]
Solution to an ODE

Suppose $\phi(t)$ solves the ODE $y'' + p(t)y' + q(t)y = 0$ then

$$0 = \phi''(t) + p(t)\phi'(t) + q(t)\phi(t)$$
$$\phi''(t) = -p(t)\phi'(t) - q(t)\phi(t)$$
$$\phi''(t_0) = -p(t_0)\phi'(t_0) - q(t_0)\phi(t_0)$$
$$2!a_2 = -p(t_0)a_1 - q(t_0)a_0.$$

Thus we may find $a_2$ in terms of $a_0$ and $a_1$. 
With patience we can find higher order terms in the series solution through *continued differentiation*.

\[
\begin{align*}
\phi''(t) & = -p(t)\phi'(t) - q(t)\phi(t) \\
\phi'''(t) & = -p'(t)\phi'(t) - p(t)\phi''(t) - q'(t)\phi(t) - q(t)\phi'(t)
\end{align*}
\]
With patience we can find higher order terms in the series solution through continued differentiation.

\[
\begin{align*}
\phi''(t) &= -p(t)\phi'(t) - q(t)\phi(t) \\
\phi'''(t) &= -p'(t)\phi'(t) - p(t)\phi''(t) - q'(t)\phi(t) - q(t)\phi'(t) \\
\phi'''(t_0) &= -p'(t_0)\phi'(t_0) - p(t_0)\phi''(t_0) - q'(t_0)\phi(t_0) - q(t_0)\phi'(t_0) \\
3!a_3 &= -p'(t_0)a_1 - 2p(t_0)a_2 - q'(t_0)a_0 - q(t_0)a_1
\end{align*}
\]
Continued Differentiation (1 of 2)

With patience we can find higher order terms in the series solution through **continued differentiation**.

\[
\phi''(t) = -p(t)\phi'(t) - q(t)\phi(t)
\]

\[
\phi'''(t) = -p'(t)\phi'(t) - p(t)\phi''(t) - q'(t)\phi(t) - q(t)\phi'(t)
\]

\[
\phi'''(t_0) = -p'(t_0)\phi'(t_0) - p(t_0)\phi''(t_0) - q'(t_0)\phi(t_0) - q(t_0)\phi'(t_0)
\]

\[
3!a_3 = -p'(t_0)a_1 - 2p(t_0)a_2 - q'(t_0)a_0 - q(t_0)a_1
\]

Substituting the previously determined value of \(a_2\) we may find \(a_3\) in terms of \(a_0\) and \(a_1\).
We can proceed by repeated differentiation to find $a_4$, $a_5$, ... provided:
We can proceed by repeated differentiation to find $a_4$, $a_5$, ...

...provided:

- $p(t)$ and $q(t)$ have derivatives of all orders, and
We can proceed by repeated differentiation to find $a_4$, $a_5$, … provided:

- $p(t)$ and $q(t)$ have derivatives of all orders, and
- we can show the resulting power series converges.
Example

Assuming that \( y = \phi(t) \) is a solution to the IVP:

\[
\begin{align*}
    y'' + t^2 y' + (\sin t)y &= 0 \\
    y(0) &= 1 \\
    y'(0) &= -1
\end{align*}
\]

find the first four nonzero terms in the power series representation of \( \phi(t) \).
Solution

\[ \phi(0) = (0!)a_0 = 1 \implies a_0 = 1 \]
\[ \phi'(0) = (1!)a_1 = -1 \implies a_1 = -1 \]
\[ \phi''(0) = (2!)a_2 = -(0^2)\phi'(0) - (\sin 0)\phi(0) = 0 \implies a_2 = 0 \]
\[ \phi'''(0) = (3!)a_3 = -(0^2)\phi''(0) - (2(0) + \sin 0)\phi'(0) - (\cos 0)\phi(0) \]
\[ = -1 \implies a_3 = -\frac{1}{3!} \]
\[ \phi^{(4)}(0) = (4!)a_4 = (\sin 0)\phi(0) - 2(1 + \cos 0)\phi'(0) \]
\[ - (4(0) + \sin 0)\phi''(0) - (0^2)\phi'''(0) \]
\[ = -4 \implies a_4 = \frac{1}{3!} \]
\[ \phi(t) = 1 - t - \frac{t^3}{6} + \frac{t^4}{6} + \cdots \]
Analytic Functions

If $p(t)$ and $q(t)$ are analytic functions at $t_0$, in other words have Taylor series expansions about $t_0$ which converge to $p(t)$ and $q(t)$ respectively then $p$ and $q$ will have derivatives of all orders at $t_0$.

\[
p(t) = \sum_{n=0}^{\infty} p_n(t - t_0)^n
\]

\[
q(t) = \sum_{n=0}^{\infty} q_n(t - t_0)^n
\]
Ordinary and Singular Points Revisited

Suppose

\[ P(t) y'' + Q(t) y' + R(t) y = 0 \]
\[ y'' + \frac{Q(t)}{P(t)} y' + \frac{R(t)}{P(t)} y = 0 \]
\[ y'' + p(t) y' + q(t) y = 0 \]

where

\[ p(t) = \frac{Q(t)}{P(t)} \quad \text{and} \quad q(t) = \frac{R(t)}{P(t)} \]

If \( p(t) \) and \( q(t) \) are analytic at \( t_0 \), then we say that \( t_0 \) is an ordinary point of the ODE. Otherwise \( t_0 \) is a singular point.
Ordinary and Singular Points Revisited

Suppose

\[ P(t)y'' + Q(t)y' + R(t)y = 0 \]

\[ y'' + \frac{Q(t)}{P(t)} y' + \frac{R(t)}{P(t)} y = 0 \]

\[ y'' + p(t)y' + q(t)y = 0 \]

where

\[ p(t) = \frac{Q(t)}{P(t)} \quad \text{and} \quad q(t) = \frac{R(t)}{P(t)} \]

If \( p(t) \) and \( q(t) \) are analytic at \( t_0 \), then we say that \( t_0 \) is an ordinary point of the ODE. Otherwise \( t_0 \) is a singular point.
Main Result

Theorem

If \( t_0 \) is an ordinary point of the ODE

\[
P(t)y'' + Q(t)y' + R(t)y = 0
\]

then the general solution of the ODE is

\[
y(t) = \sum_{n=0}^{\infty} a_n(t - t_0)^n = a_0 y_1(t) + a_1 y_2(t)
\]

where \( a_0 \) and \( a_1 \) are arbitrary and \( y_1 \) and \( y_2 \) are linearly independent series solutions that are analytic at \( t_0 \). Further the radius of convergence for each of \( y_1 \) and \( y_2 \) is at least as large as the minimum of the radii of convergence for the series \( p(t) = Q(t)/P(t) \) and \( q(t) = R(t)/P(t) \).
Radius of Convergence

If \( p(t) = Q(t)/P(t) \) and \( q(t) = R(t)/P(t) \) and \( p(t) \) and \( q(t) \) are analytic at \( t_0 \) then from the theory of complex variables we have the result that

the radius of convergence of \( p(t) \) (and similarly \( q(t) \)) is at least as large as the minimum distance from \( t_0 \) to any root of \( P(t) \) in the complex plane.
Example

The value $t_0 = 1$ is an ordinary point of the ODE

$$t^2 y'' + (1 + t)y' + 3(\ln t)y = 0.$$ 

Find the radius of convergence of $p(t) = \frac{1 + t}{t^2}$ and $q(t) = \frac{3 \ln t}{t^2}$. 

Example

The value \( t_0 = 0 \) is an ordinary point of the ODE

\[(1 + t^4)y'' + 4ty' + y = 0.\]

Find the radius of convergence of \( p(t) = \frac{4t}{1 + t^4} \) and

\[q(t) = \frac{1}{1 + t^4}.\]
Example

The value \( t_0 = 0 \) is an ordinary point of the ODE

\[
(1 + t^4)y'' + 4ty' + y = 0.
\]

Find the radius of convergence of \( p(t) = \frac{4t}{1 + t^4} \) and \( q(t) = \frac{1}{1 + t^4} \).

Using Euler's Identity:

\[
t^4 + 1 = 0
\]

\[
t^4 = -1 = e^{i(2n+1)\pi}
\]

\[
t = e^{i(2n+1)\pi/4}
\]

\[
t = \pm \frac{\sqrt{2}}{2} \pm i \frac{\sqrt{2}}{2}.
\]
Illustration
The value $t_0 = 1$ is an ordinary point of the ODE

$$(1 + t^4)y'' + 4ty' + y = 0.$$ 

Find the radius of convergence of $p(t) = \frac{4t}{1 + t^4}$ and $q(t) = \frac{1}{1 + t^4}$. 
Homework

- Read Section 5.3
- Exercises: 1–7 odd, 10, 11, 22–29