Regular Singular Points
MATH 365 Ordinary Differential Equations

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Consider the generic second order linear homogeneous ODE:

\[ P(t) y'' + Q(t) y' + R(t) y = 0. \]

- If we assume that \( P, Q, \) and \( R \) are polynomials and have no factors in common then the **singular points** of the ODE are the values of \( t \) for which \( P(t) = 0 \).
- If \( P, Q, \) or \( R \) is non-polynomial then a singular point is a value of \( t_0 \) for which \( P(t_0) = 0 \) and at least one of \( Q(t_0) \) or \( R(t_0) \) is nonzero.
Examples

Find all the singular points of the following ODEs.

- \( t^2 (1 - t^2) y'' + \frac{2}{t} y' + 4y = 0 \)
- \( t (1 - t^2)^3 y'' + (1 - t^2)^2 y' + 2(1 + t)y = 0 \)
- \( t^2 y'' + 2(e^t - 1)y' + (e^{-t} \cos t)y = 0 \)
$t^2(1 - t^2)y'' + \frac{2}{t}y' + 4y = 0$

Let

$P(t) = t^2(1 - t^2) = t^2(1 - t)(1 + t)$

$Q(t) = \frac{2}{t}$

$R(t) = 4$

then the singular points are $t = 0, -1, 1$. 
Solutions (2 of 3)

\[ t(1 - t^2)^3 y'' + (1 - t^2)^2 y' + 2(1 + t)y = 0 \]

Let

\begin{align*}
P(t) &= t(1 - t^2)^3 = t(1 - t)^3(1 + t)^3 \\
Q(t) &= (1 - t^2)^2 = (1 - t)^2(1 + t)^2 \\
R(t) &= 2(1 + t)
\end{align*}

then the singular points are \( t = 0, 1 \).

**Question**: why is \( t = -1 \) not a singular point?
Solutions (3 of 3)

\[ t^2 y'' + 2(e^t - 1)y' + (e^{-t} \cos t)y = 0 \]

Let

\[ P(t) = t^2 \]
\[ Q(t) = 2(e^t - 1) \]
\[ R(t) = e^{-t} \cos t \]

then the singular point is \( t = 0 \).
Behavior of Solutions Near Singular Points

Even if \( t_0 \) is a singular point, we may want to study the solution to the ODE near \( t_0 \).

Of particular importance is the behavior of solutions \( y_1(t) \) and \( y_2(t) \) as \( t \to t_0 \).
Even if $t_0$ is a singular point, we may want to study the solution to the ODE near $t_0$.

Of particular importance is the behavior of solutions $y_1(t)$ and $y_2(t)$ as $t \to t_0$.

- $\lim_{t \to t_0} |y_1(t)| < \infty$ and $\lim_{t \to t_0} |y_2(t)| < \infty$
- $\lim_{t \to t_0} |y_1(t)| = \infty$ and $\lim_{t \to t_0} |y_2(t)| < \infty$
- $\lim_{t \to t_0} |y_1(t)| = \infty$ and $\lim_{t \to t_0} |y_2(t)| = \infty$
Example (1 of 4)

\[ t^2 y'' - 3ty' + 4y = 0 \]

\[ y_1(t) = t^2 \]

\[ y_2(t) = t^2 \ln |t| \]
Example (2 of 4)

\[
t^2 y'' + 3ty' + 5y = 0
\]

\[
y_1(t) = t^{-1} \cos(2 \ln |t|)
\]

\[
y_2(t) = t^{-1} \sin(2 \ln |t|)
\]
Example (3 of 4)

\[ t^2 y'' + 4ty' + 2y = 0 \]

\[ y_1(t) = t^{-1} \]

\[ y_2(t) = t^{-2} \]
Example (4 of 4)

\[ t(t - 3)y'' + (t + 1)y' - 2y = 0 \]
To extend the power series solution technique to singular points we must place some restrictions on the ODE

\[ P(t)y'' + Q(t)y' + R(t)y = 0. \]

If \( t_0 \) is a singular point, it may only be a “weak singularity”.
Regular Singular Points

Definition
Given the ODE

\[ P(t)y'' + Q(t)y' + R(t)y = 0. \]

for which \( t_0 \) is a singular point, we say \( t_0 \) is a **regular singular point** if both functions

\[ (t - t_0) \frac{Q(t)}{P(t)} \quad \text{and} \quad (t - t_0)^2 \frac{R(t)}{P(t)} \]

are analytic (in other words have convergent Taylor series centered at \( t_0 \)).
Remarks

- The analytic requirement is satisfied for polynomial $P$, $Q$, and $R$ if the following limits are both finite.

\[
\lim_{t \to t_0} (t - t_0) \frac{Q(t)}{P(t)} = p_0 \quad \text{and} \quad \lim_{t \to t_0} (t - t_0)^2 \frac{R(t)}{P(t)} = q_0
\]

- When $P$, $Q$, and $R$ are polynomials this means that for $t$ near $t_0$

\[
\frac{Q(t)}{P(t)} \propto \frac{1}{t - t_0} \quad \text{and} \quad \frac{R(t)}{P(t)} \propto \frac{1}{(t - t_0)^2}.
\]

- any singular point which is not regular is called an **irregular singular point**.
Examples

Determine the regular and irregular singular points for the following ODEs.

- \[ t^2(1 - t^2)y'' + \frac{2}{t}y' + 4y = 0 \]
- \[ t(1 - t^2)^3y'' + (1 - t^2)^2y' + 2(1 + t)y = 0 \]
- \[ t^2y'' + 2(e^t - 1)y' + (e^{-t} \cos t)y = 0 \]
- \[ t(t - 3)y'' + (t + 1)y' - 2y = 0 \]
Solution (1 of 4)

\[ t^2(1 - t^2)y'' + \frac{2}{t}y' + 4y = 0 \]

\[ t_0 = -1 \]
\[ t_0 = 0 \]
\[ t_0 = 1 \]

\[ \lim_{t \to 1} (t - 1) \frac{2/t}{t^2(1 - t^2)} = -1 \]
\[ \lim_{t \to 1} (t - 1)^2 \frac{4}{t^2(1 - t^2)} = 0 \]
\[ \lim_{t \to -1} (t + 1) \frac{2/t}{t^2(1 - t^2)} = -1 \]
\[ \lim_{t \to -1} (t + 1)^2 \frac{4}{t^2(1 - t^2)} = 0 \]
\[ \lim_{t \to 0} t \frac{2/t}{t^2(1 - t^2)} = \infty \]
\[ \lim_{t \to 0} t^2 \frac{4}{t^2(1 - t^2)} = 4 \]

Values \( t_0 = \pm 1 \) are regular singular points, while \( t_0 = 0 \) is an irregular singular point.
$$t(1 - t^2)^3 y'' + (1 - t^2)^2 y' + 2(1 + t)y = 0$$

$$t_0 = 0$$

$$t_0 = 1$$

$$\lim_{t \to 0} t \frac{(1 - t^2)^2}{t(1 - t^2)^3} = 1$$

$$\lim_{t \to 1} (t - 1) \frac{(1 - t^2)^2}{t(1 - t^2)^3} = -\frac{1}{2}$$

$$\lim_{t \to 0} t^2 \frac{2(1 + t)}{t(1 - t^2)^3} = 0$$

$$\lim_{t \to 1} (t - 1)^2 \frac{2(1 + t)}{t(1 - t^2)^3} = -\infty$$

Value $t_0 = 0$ is a regular singular point, while $t_0 = 1$ is an irregular singular point.
Solution (3 of 4)

\[ t^2 y'' + 2(e^t - 1)y' + (e^{-t} \cos t)y = 0 \]
\[ t_0 = 0 \]

\[ \lim_{t \to 0} t \frac{2(e^t - 1)}{t^2} = 2 \]
\[ \lim_{t \to 0} t^2 \frac{e^{-t} \cos t}{t^2} = 1 \]

Value \( t_0 = 0 \) is a regular singular point.
Solution (4 of 4)

\[ t(t - 3)y'' + (t + 1)y' - 2y = 0 \]

\[
t_0 = 0
\]

\[
t_0 = 3
\]

\[
\lim_{t \to 0} t \frac{t + 1}{t(t - 3)} = -\frac{1}{3}
\]

\[
\lim_{t \to 3} (t - 3) \frac{t + 1}{t(t - 3)} = \frac{4}{3}
\]

\[
\lim_{t \to 0} t^2 \frac{-2}{t(t - 3)} = 0
\]

\[
\lim_{t \to 3} (t - 3)^2 \frac{-2}{t(t - 3)} = 0
\]

Values \( t_0 = 0 \) and \( t_0 = 3 \) are regular singular points.
Consider the following ODE for which \( t = 0 \) is a regular singular point.

\[
t^2 y'' + 2t y' - (1 + t)y = 0
\]

Assuming the solution is of the form \( y(t) = \sum_{n=0}^{\infty} a_n t^n \) find the coefficients \( a_n \).
Solution (1 of 2)

\[
0 = t^2 \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + 2t \sum_{n=1}^{\infty} na_n t^{n-1} - (1 + t) \sum_{n=0}^{\infty} a_n t^n
\]

\[
= \sum_{n=0}^{\infty} n(n-1)a_n t^n + \sum_{n=0}^{\infty} 2na_n t^n - \sum_{n=0}^{\infty} a_n t^n - \sum_{n=0}^{\infty} a_n t^{n+1}
\]

\[
= \sum_{n=0}^{\infty} [n(n-1) + 2n - 1]a_n t^n - \sum_{n=1}^{\infty} a_{n-1} t^n
\]

\[
= -a_0 + \sum_{n=1}^{\infty} [n(n-1) + 2n - 1]a_n t^n - \sum_{n=1}^{\infty} a_{n-1} t^n
\]

\[
0 = -a_0 + \sum_{n=1}^{\infty} [(n(n-1) + 2n - 1)a_n - a_{n-1}] t^n
\]
Solution (2 of 2)

\[ 0 = -a_0 + \sum_{n=1}^{\infty} [(n(n-1) + 2n - 1)a_n - a_{n-1}]t^n \]

\[ a_0 = 0 \]

\[ a_n = \frac{a_{n-1}}{n(n-1) + 2n - 1} \quad \text{(for } n \geq 1) \]
Solution (2 of 2)

\[0 = -a_0 + \sum_{n=1}^{\infty} [(n(n-1) + 2n - 1)a_n - a_{n-1}]t^n\]

\[a_0 = 0\]
\[a_n = \frac{a_{n-1}}{n(n-1) + 2n - 1} \quad \text{(for } n \geq 1)\]
\[a_1 = 0\]
\[a_2 = 0\]
\[a_3 = 0\]
\[\vdots\]
Homework

- Read Section 5.4
- Exercises: 17–39 odd