Second Order Linear Equations

MATH 365, Ordinary Differential Equations

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Introduction

General second order form of an ODE:

\[ \frac{d^2 y}{dt^2} = f(t, y, \frac{dy}{dt}) \]

If the second order equation is of the \textit{linear} type then it can always be written as follows.

\[ y'' + p(t) y' + q(t) y = g(t) \]

Some textbooks and authors write the general form as below.

\[ P(t) y'' + Q(t) y' + R(t) y = G(t) \]

This form is equivalent to the one above if we make the following assignments.

\[ p(t) = \frac{Q(t)}{P(t)} \]
\[ q(t) = \frac{R(t)}{P(t)} \]
\[ g(t) = \frac{G(t)}{P(t)} \]

If \( g(t) = 0 \) then we may say the second order linear equation is \textit{homogeneous}. If \( p(t) \) and \( q(t) \) are constants then we call the second order linear equation, \textit{constant coefficient}. Today we will learn the general technique for solving second order linear, constant coefficient, homogeneous ODEs and IVPs.

Applications

Second order linear ODEs/IVPs frequently arise in models of fluid motion, electromagnetism, heat conduction, wave motion, and vibration.

The general mathematical model of the motion of a mass connected to a Hooke’s Law spring, subject to viscous damping, and an external force is below.

\[ m y'' + \gamma y' + k y = F(t) \]

Bessel’s equation must be solved in order to model the motion of a vibrating circular membrane (for example a drum).
Legendre’s equation and its solution are often encountered in quantum mechanics.

\[(1 - t^2) y'' - 2t y' + \alpha (\alpha + 1) y = 0\]

### Motivating Examples

- **Example (1 of 2)**

  Consider the second order linear, constant coefficient, homogeneous ODE:

  \[y'' - y = 0\]

  Check that \(y(t) = c_1 e^t + c_2 e^{-t}\) is a solution to this ODE. The expressions \(c_1\) and \(c_2\) are arbitrary constants.

- **Example (2 of 2)**

  Consider the second order linear, constant coefficient, homogeneous IVP:

  \[y'' - y = 0, \ y(0) = 2, \ y'(0) = -1\]

  Find the solution to this IVP.

### General Solution to Constant Coefficient ODEs/IVPs

Now consider the case the general second order, linear, constant coefficient, homogeneous ODE below.

\[a y'' + b y' + c y = 0\]

We will assume that \(y(t) = e^{rt}\) where \(r\) is a constant. Substitute this solution into the ODE and derive the characteristic equation:

\[a r^2 + b r + c = 0\]

If \(r\) is a solution to the characteristic equation then \(y(t) = e^{rt}\) is a solution to the ODE.

### Examples

- **Example (1 of 3)**

  Solve the following second order, linear, constant coefficient, homogeneous IVP.

  \[y'' + 3 y' + 2 y = 0, \ y(0) = 2, \ y'(0) = 0\]
Example (2 of 3)

Solve the following second order, linear, constant coefficient, homogeneous IVP.

\[ y'' + y' - 20y = 0, \quad y(2) = 1, \quad y'(2) = 0 \]

\[
y[t_] := -2 \exp[-2t] + 4 \exp[-t]; Plot[Tooltip[{y[t], y'[t]}], \{t, 0, 5\}, AxesLabel -> {"t", ""}]\]

Example (3 of 3)

Solve the following second order, linear, constant coefficient, homogeneous IVP.

\[ 3y'' - 12y' - 36y = 0, \quad y(0) = 2, \quad y'(0) = 1 \]

\[
y[t_] := \left(\frac{4}{9}\right) \exp[5(2 - t)] + \left(\frac{5}{9}\right) \exp[4(t - 2)];\]

\[
Plot[Tooltip[{y[t], y'[t]}], \{t, 1.8, 2.2\}, AxesLabel -> {"t", ""}]\]
\begin{align*}
y[t_] &= (5/8) \exp[6 t] + (11/8) \exp[-2 t]; \\
&\quad \text{Plot[Tooltip[{y[t], y'[t]}], \{t, 0, 1/5\}, AxesLabel \rightarrow \{"t", "\}, PlotRange \rightarrow \{0, 10\}]}
\end{align*}

\textbf{Homework}

Read Section 3.1

Pages 142–143: work exercises 1–27 odd