Millersville University **Name** ________________________________
Mathematics Department
MATH 365, *Ordinary Differential Equations*, Final Examination
December 11, 2008, 10:15AM-12:15PM

Please answer the following questions. Show all work and write neatly. Answers without justifying work will receive no credit. Partial credit will be given as appropriate, do not leave any problem blank. The point values of problems are indicated in parentheses.

1. (8 points each) Solve the following ordinary differential equations and initial value problems.

   (a) $x\sqrt{1+y^2}\,dx = y\sqrt{1+x^2}\,dy; \quad y(0) = 0$

$$\begin{aligned}
x\sqrt{1+y^2}\,dx &= y\sqrt{1+x^2}\,dy \\
\frac{x}{\sqrt{1+x^2}}\,dx &= \frac{y}{\sqrt{1+y^2}}\,dy \\
\int_0^x \frac{t}{\sqrt{1+t^2}}\,dt &= \int_0^y \frac{t}{\sqrt{1+t^2}}\,dt \\
\left(\sqrt{1+t^2}\right)\Big|_0^x &= \left(\sqrt{1+t^2}\right)\Big|_0^y \\
\sqrt{1+x^2} - 1 &= \sqrt{1+y^2} - 1 \\
x^2 &= y^2
\end{aligned}$$

(b) $y'' - 4y = 4e^{2t}$

The characteristic equation for the corresponding homogeneous equation is

$$r^2 - 4 = 0$$

which implies $r_1 = 2$ and $r_2 = -2$. Thus the complementary solution is

$$y_c(t) = c_1 e^{2t} + c_2 e^{-2t}.$$

The particular solution to the nonhomogeneous equation has the form

$$\begin{aligned} Y(t) &= Ate^{2t} \\ Y'(t) &= A(1+2t)e^{2t} \\ Y''(t) &= 4A(1+t)e^{2t} \end{aligned}$$

thus upon substituting into the nonhomogeneous equation we find that

$$\begin{aligned} 4A(1+t)e^{2t} - 4Ate^{2t} &= 4e^{2t} \\ A(1+t) - At &= 1 \\ A &= 1. \end{aligned}$$

Thus the general solution to the nonhomogeneous equation is

$$y(t) = y_c(t) + Y(t) = c_1 e^{2t} + c_2 e^{-2t} + te^{2t}.$$

(c) $x^2y'' + xy' - 9y = \sqrt{x} + \dfrac{1}{\sqrt{x}}, \quad x > 0$

If we make the change of variable $z = \ln x$ in this Euler equation then the corresponding transformed homogeneous equation takes the form

$$\frac{d^2y}{dz^2} - 9y = 0$$

which has characteristic equation

$$r^2 - 9 = 0$$

and thus $r_1 = 3$ and $r_2 = -3$. Thus the complementary solution to the original equation is

$$y_c(x) = c_1x^3 + c_2x^{-3}.$$

The Wronskian of these solutions is

$$W(x) = \begin{vmatrix} x^3 & x^{-3} \\ 3x^2 & -3x^{-4} \end{vmatrix} = -6x^{-1}.$$

We will use the method of variation of parameters to find the particular solution. We must re-write the ODE in the form below.

$$y'' + \frac{1}{x}y' - \frac{9}{x^2}y = x^{-3/2} + x^{-5/2}$$

Applying the variation of parameters formula we find

$$\begin{aligned} \mu_1'(x) &= -\frac{(x^{-3/2} + x^{-5/2})x^{-3}}{-6x^{-1}} \\ &= \frac{1}{6}(x^{-7/2} + x^{-9/2}) \\ \mu_1(x) &= -\frac{1}{3}\left(\frac{1}{5}x^{-5/2} + \frac{1}{7}x^{-7/2}\right) \end{aligned}$$

and

$$\begin{aligned} \mu_2'(x) &= \frac{(x^{-3/2} + x^{-5/2})x^{3}}{-6x^{-1}} \\ &= -\frac{1}{6}(x^{5/2} + x^{3/2}) \\ \mu_2(x) &= -\frac{1}{3}\left(\frac{1}{7}x^{7/2} + \frac{1}{5}x^{5/2}\right). \end{aligned}$$

Thus the general solution to the nonhomogeneous Euler equation is

$$\begin{aligned} y(x) &= y_c(x) + Y(x) \\ &= c_1x^3 + c_2x^{-3} - \frac{1}{3}\left(\frac{1}{5}x^{1/2} + \frac{1}{7}x^{-1/2}\right) - \frac{1}{3}\left(\frac{1}{7}x^{1/2} + \frac{1}{5}x^{-1/2}\right) \\ &= c_1x^3 + c_2x^{-3} - \frac{4}{35}x^{1/2} - \frac{4}{35}x^{-1/2}. \end{aligned}$$

(d) $(x - y\cos x)\,dx - (\sin x + y)\,dy = 0$

Since

$$\frac{\partial}{\partial y}(x - y\cos x) = -\cos x = \frac{\partial}{\partial x}(-\sin x - y)$$

the ordinary differential equation is exact. The implicit form of the solution is

$$\begin{aligned}
\psi(x,y) &= \int (x - y\cos x)\,dx \\
&= \frac{1}{2}x^2 - y\sin x + h(y) \\
\frac{\partial}{\partial y}\psi(x,y) &= \frac{\partial}{\partial y}\left(\frac{1}{2}x^2 - y\sin x + h(y)\right) \\
-\sin x - y &= -\sin x + h'(y) \\
h'(y) &= -y \\
h(y) &= -\frac{1}{2}y^2.
\end{aligned}$$

Thus

$$\frac{1}{2}x^2 - y\sin x - \frac{1}{2}y^2 = C.$$

(e) $y'' + 3y' + 2y = \delta(t-2) + u_3(t); \quad y(0) = y'(0) = 0$
The functions δ and u refer to the Dirac delta function and unit step functions respectively.

$$\begin{aligned}
\mathcal{L}\{y'' + 3y' + 2y\} &= \mathcal{L}\{\delta(t-2) + u_3(t)\} \\
s^2Y(s) + 3sY(s) + 2Y(s) &= e^{-2s} + \frac{e^{-3s}}{s} \\
Y(s)(s^2 + 3s + 2) &= e^{-2s} + \frac{e^{-3s}}{s} \\
Y(s) &= \frac{e^{-2s}}{s^2 + 3s + 2} + \frac{e^{-3s}}{s(s^2 + 3s + 2)} \\
&= \frac{e^{-2s}}{(s+1)(s+2)} + \frac{e^{-3s}}{s(s+1)(s+2)}
\end{aligned}$$

Note that

$$\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

and

$$\frac{1}{s(s+1)(s+2)} = \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}.$$

Therefore

$$\begin{aligned}
Y(s) &= \frac{e^{-2s}}{s+1} - \frac{e^{-2s}}{s+2} + \frac{1}{2}\frac{e^{-3s}}{s} - \frac{e^{-3s}}{s+1} + \frac{1}{2}\frac{e^{-3s}}{s+2} \\
y(t) &= u_2(t)(e^{-(t-2)} - e^{-2(t-2)}) + u_3(t)\left(\frac{1}{2} - e^{-(t-3)} + \frac{1}{2}e^{-2(t-3)}\right)
\end{aligned}$$

2. (6 points) Deer hunting season begins November 1st and lasts for thirty days. Hunters kill 450 deer per month during the season. The deer population grows exponentially at a rate of 0.057/month. If there are 500 deer in an isolated region (where no deer can leave or enter) on January 1st, how many deer will be in the region on the following January 1st?

A mathematical model for the deer population is one of exponential growth subject to a discontinuous forcing function.

$$\begin{aligned} p' &= 0.057p - 450(u_{10}(t) - u_{11}(t)) \\ p' - \frac{57}{1000}p &= -450(u_{10}(t) - u_{11}(t)) \end{aligned}$$

Taking the Laplace transform of both sides and using the initial population $p(0) = 500$ we get

$$\begin{aligned} sP(s) - 500 - \frac{57}{1000}P(s) &= -450\left(\frac{e^{-10s}}{s} - \frac{e^{-11s}}{s}\right) \\ \left(s - \frac{57}{1000}\right)P(s) &= 500 - 450\left(\frac{e^{-10s}}{s} - \frac{e^{-11s}}{s}\right) \\ P(s) &= \frac{500}{s - \frac{57}{1000}} - \frac{450}{s(s - \frac{57}{1000})}\left(e^{-10s} - e^{-11s}\right) \\ &= \frac{500}{s - \frac{57}{1000}} - 450\left(-\frac{1000}{57s} + \frac{1000}{57(s - \frac{57}{1000})}\right)\left(e^{-10s} - e^{-11s}\right) \\ &= \frac{500}{s - \frac{57}{1000}} - \frac{450000}{57}\left(\frac{1}{s - \frac{57}{1000}} - \frac{1}{s}\right)\left(e^{-10s} - e^{-11s}\right) \\ p(t) &= 500e^{0.057t} - \frac{450000}{57}u_{10}(t)(e^{0.057(t-10)} - 1) \\ &\quad + \frac{450000}{57}u_{11}(t)(e^{0.057(t-11)} - 1) \end{aligned}$$

Thus

$$\begin{aligned} p(12) &= 500e^{0.057(12)} - \frac{450000}{57}u_{10}(12)(e^{0.057(12-10)} - 1) + \frac{450000}{57}u_{11}(12)(e^{0.057(12-11)} - 1) \\ &= 500e^{0.684} - \frac{450000}{57}(e^{0.114} - 1) + \frac{450000}{57}(e^{0.057} - 1) \\ &= 500e^{0.684} - \frac{450000}{57}e^{0.114} + \frac{450000}{57}e^{0.057} \\ &= 500.660 \approx 501. \end{aligned}$$

3. (12 points) Find two linearly independent solutions to the differential equation below **using a power series expansion** about $x_0 = 0$. State the **recurrence relation** and find a formula for the **general term** of each power series.

$$y'' + y = 0$$

The point $x_0 = 0$ is an ordinary point for the ODE. If we assume $y(t) = \sum_{n=0}^{\infty} a_n t^n$ then

$$\begin{aligned} 0 &= \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=0}^{\infty} a_n t^n \\ &= \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2} t^n + \sum_{n=0}^{\infty} a_n t^n \\ &= \sum_{n=0}^{\infty} \left[(n+1)(n+2)a_{n+2} + a_n\right] t^n. \end{aligned}$$

The recurrence relation is $a_{n+2} = -\dfrac{a_n}{(n+2)(n+1)}$.

If we assume $a_0 = 1$ and $a_1 = 0$ then $a_{2n+1} = 0$ for $n = 0, 1, \ldots$ and

$$\begin{aligned} a_2 &= -\frac{1}{2!} \\ a_4 &= \frac{1}{4!} \\ &\vdots \\ a_{2n} &= \frac{(-1)^n}{(2n)!}. \end{aligned}$$

Thus $y_1(t) = \sum_{n=0}^{\infty} \dfrac{(-1)^n t^{2n}}{(2n)!} = \cos t$.

If we assume $a_0 = 0$ and $a_1 = 1$ then $a_{2n} = 0$ for $n = 0, 1, \ldots$ and

$$\begin{aligned} a_3 &= -\frac{1}{3!} \\ a_5 &= \frac{1}{5!} \\ &\vdots \\ a_{2n+1} &= \frac{(-1)^n}{(2n+1)!}. \end{aligned}$$

Thus $y_2(t) = \sum_{n=0}^{\infty} \dfrac{(-1)^n t^{2n+1}}{(2n+1)!} = \sin t$.

4. A 256 lb weight is suspended from a vertical spring having spring constant of 200 lb/ft. The weight is raised 3 inches above its equilibrium position and released.

   (a) (6 points) Find the displacement $u(t)$ of the mass for all $t > 0$.
   The displacement will be the solution to the following IVP.

$$\begin{aligned} 8u'' + 200u &= 0 \\ u(0) &= -\frac{1}{4} \\ u'(0) &= 0 \end{aligned}$$

   The general solution to the ODE is $u(t) = c_1 \cos 5t + c_2 \sin 5t$. Making use of the initial conditions we may evaluate c_1 and c_2 and find that

$$u(t) = -\frac{1}{4} \cos 5t.$$

   (b) (4 points) Find the period and amplitude of the vibration.

$$\begin{aligned} \text{period} &= \frac{2\pi}{5} \\ \text{amplitude} &= \frac{1}{4} \end{aligned}$$

   (c) (4 points) Find the position of the mass at a time $t = \pi/3$ after release and determine the direction and speed of the mass.

$$\begin{aligned} u(\pi/3) &= -\frac{1}{4} \cos(5\pi/3) = -\frac{1}{8} \\ u'(\pi/3) &= \frac{5}{4} \sin(5\pi/3) = -\frac{5\sqrt{3}}{8} \end{aligned}$$

   The mass is 1/8 feet above its equilibrium position and traveling upward at $5\sqrt{3}/8$ feet per second.

5. Consider the ordinary differential equation,

$$4xy'' + 2y' + y = 0.$$

(a) (2 points) Show that $x = 0$ is a regular singular point for this equation.

Let $P(x) = 4x$, $Q(x) = 2$, and $R(x) = 1$, then since $P(0) = 0$, $x = 0$ is a singular point.

$$\begin{aligned} \lim_{x \to 0} x\left(\frac{2}{4x}\right) &= \frac{1}{2} = p_0 \\ \lim_{x \to 0} x^2\left(\frac{1}{4x}\right) &= 0 = q_0 \end{aligned}$$

Since the limits above exist and are finite $x = 0$ is a regular singular point.

(b) (4 points) Find the exponents of singularity corresponding to the regular singular point $x = 0$.

The exponents of singularity are the roots of the indicial equation:

$$\begin{aligned} 0 &= r(r-1) + p_0 r + q_0 \\ &= r(r-1) + \frac{1}{2}r \\ &= r\left(r - \frac{1}{2}\right). \end{aligned}$$

Thus $r_1 = \dfrac{1}{2}$ and $r_2 = 0$.

(c) (6 points) Find a series solution to the equation corresponding to the larger of the exponents of singularity. Write out at least the first four terms of the series.

We will assume $y(x) = \sum_{n=0}^{\infty} a_n x^{r+n}$, then

$$\begin{aligned}
0 &= 4x\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{r+n-2} + 2\sum_{n=0}^{\infty}(r+n)a_n x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n} \\
&= \sum_{n=0}^{\infty} 4(r+n)(r+n-1)a_n x^{r+n-1} + \sum_{n=0}^{\infty} 2(r+n)a_n x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n} \\
&= \sum_{n=0}^{\infty} 2(r+n)\left[2(r+n-1)+1\right] a_n x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n} \\
&= \sum_{n=0}^{\infty} 2(r+n)(2r+2n-1)a_n x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n} \\
&= \sum_{n=0}^{\infty} 2(r+n)(2r+2n-1)a_n x^{r+n-1} + \sum_{n=1}^{\infty} a_{n-1} x^{r+n-1} \\
&= 2r(2r-1)a_0 x^{r-1} + \sum_{n=1}^{\infty} 2(r+n)(2r+2n-1)a_n x^{r+n-1} + \sum_{n=1}^{\infty} a_{n-1} x^{r+n-1} \\
&= 2r(2r-1)a_0 x^{r-1} + \sum_{n=1}^{\infty} \left[2(r+n)(2r+2n-1)a_n + a_{n-1}\right] x^{r+n-1}.
\end{aligned}$$

The recurrence relation takes the form

$$\begin{aligned}
a_n &= -\frac{a_{n-1}}{(2r+2n)(2r+2n-1)} \\
&= -\frac{a_{n-1}}{2n(2n+1)}.
\end{aligned}$$

Hence if we assume $a_0 = 1$ then

$$\begin{aligned}
a_1 &= -\frac{1}{3!} \\
a_2 &= \frac{1}{5!} \\
a_3 &= -\frac{1}{7!} \\
&\vdots \\
a_n &= \frac{(-1)^n}{(2n+1)!}.
\end{aligned}$$

Therefore a solution to the ODE is

$$y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1/2}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n (\sqrt{x})^{2n+1}}{(2n+1)!} = \sin\sqrt{x}.$$

6. (8 points) Are the functions e^{3x} and xe^{3x} linearly independent on the real number line? You must justify your answer.

   Consider the Wronskian of the two functions.

$$\begin{aligned} W\left(e^{3x}, xe^{3x}\right) &= \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & (1+3x)e^{3x} \end{vmatrix} \\ &= (1+3x)e^{6x} - 3xe^{6x} \\ &= e^{6x} \neq 0 \end{aligned}$$

   for $x \in \mathbb{R}$. Thus the functions are linearly independent on $\mathbb{R}$.

7. (8 points) Using the Convolution Theorem find the inverse Laplace transform of the following function. You must express the result in the form of a convolution integral.

$$Y(s) = \frac{s^2}{s^4 - 1}$$

Since

$$Y(s) = \left(\frac{s}{s^2+1}\right)\left(\frac{s}{s^2-1}\right)$$

then

$$y(t) = \int_0^t \cos(t-\tau)\,\cosh(\tau)\,d\tau,$$

or

$$y(t) = \int_0^t \cos(\tau)\,\cosh(t-\tau)\,d\tau.$$