Millersville University **Name** ______**Answer Key**______
Department of Mathematics
MATH 365, *Ordinary Differential Equations*, Test 1
September 24, 2008

Please answer the following questions. Show all work and write neatly. Answers without justifying work will receive no credit. Partial credit will be given as appropriate, do not leave any problem blank. The point values of problems are indicated in parentheses.

1. (12 points) The human liver can be treated as a tank with a volume of 500 cm^3. A drug is entering the liver via a blood vessel. Blood flows to the liver at a rate of 10 cm^3 per second. The drug in the incoming blood has a concentration of 0.08 g/cm^3. Blood flows from the liver at a rate of 10 cm^3 per second and the concentration of drug in the outgoing blood equals the well-mixed concentration of drug in the liver. Assuming the liver contained none of the drug initially, find the concentration of the drug in the liver after 120 seconds.

   t**:** time measured in seconds

   V**:** volume of the liver ($V = 500$ cm^3)

   A**:** amount of drug in the liver measured in grams

   C**:** concentration of drug in liver ($C = A/V$)

   r**:** flow rate ($r = 10$ cm^3/sec)

   γ**:** concentration of drug in incoming blood ($\gamma = 0.08$ g/cm^3)

$$\begin{aligned}
\frac{dA}{dt} &= r\gamma - r\frac{A}{V} \\
\frac{1}{V}\frac{dA}{dt} &= \frac{r\gamma}{V} - r\frac{A}{V^2} \\
\frac{dC}{dt} &= \frac{r\gamma}{V} - \frac{r}{V}C \\
\frac{dC}{dt} + \frac{r}{V}C &= \frac{r\gamma}{V}
\end{aligned}$$

This ODE is first order and linear, an integrating factor is $\mu(t) = e^{rt/V}$.

$$\begin{aligned}
e^{rt/V}\left(\frac{dC}{dt} + \frac{r}{V}C\right) &= \frac{r\gamma}{V}e^{rt/V} \\
\frac{d}{dt}\left(e^{rt/V}C(t)\right) &= \frac{r\gamma}{V}e^{rt/V} \\
\int_0^t d\left(e^{rs/V}C(s)\right) &= \int_0^t \frac{r\gamma}{V}e^{rs/V}\,ds
\end{aligned}$$

$$
\begin{aligned}
e^{rt/V}C(t) - C(0) &= \gamma e^{rt/V} - \gamma \\
C(t) &= \gamma - \gamma e^{-rt/V} \\
C(120) &= 0.08 - 0.08e^{-10(120)/500} \\
&\approx 0.072743\text{g/cm}^3
\end{aligned}
$$

2. (10 points each) Solve each of the following ODEs or IVPs.

(a) $t^2y^2\frac{dy}{dt} = (t^2+1)(y^3-1)$

This ODE is separable.

$$\begin{aligned} t^2y^2\frac{dy}{dt} &= (t^2+1)(y^3-1) \\ \frac{y^2}{y^3-1}\,dy &= \left(1+\frac{1}{t^2}\right)dt \\ \int \frac{y^2}{y^3-1}\,dy &= \int \left(1+\frac{1}{t^2}\right)dt \\ \frac{1}{3}\ln|y^3-1| &= t-\frac{1}{t}+C \end{aligned}$$

(b) $t\frac{dy}{dt}+y=t^2, \quad y(1)=2.$

This ODE is first order linear.

$$\begin{aligned} t\frac{dy}{dt}+y &= t^2 \\ \frac{dy}{dt}+\frac{1}{t}y &= t \end{aligned}$$

An integrating factor is $\mu(t) = e^{\int \frac{1}{t}\,dt} = t$.

$$\begin{aligned} t\left(\frac{dy}{dt}+\frac{1}{t}y\right) &= t^2 \\ \frac{d}{dt}(ty) &= t^2 \\ \int_1^t d(sy(s)) &= \int_1^t s^2\,ds \\ ty(t)-y(1) &= \frac{1}{3}t^3-\frac{1}{3} \\ ty(t)-2 &= \frac{1}{3}t^3-\frac{1}{3} \\ ty(t) &= \frac{1}{3}t^3+\frac{5}{3} \\ y(t) &= \frac{1}{3}t^2+\frac{5}{3t} \end{aligned}$$

(c) $\frac{dy}{dt} = \frac{t}{y}e^{t^2-y^2}$

This ODE is separable.

$$\begin{aligned}
\frac{dy}{dt} &= \frac{t}{y}e^{t^2-y^2} \\
ye^{y^2}\,dy &= te^{t^2}\,dt \\
\int ye^{y^2}\,dy &= \int te^{t^2}\,dt \\
\frac{1}{2}e^{y^2} &= \frac{1}{2}e^{t^2} + C \\
e^{y^2} &= e^{t^2} + C
\end{aligned}$$

(d) $(\sin y)\,dt + (t\cos y - y)\,dy = 0$

Since

$$\frac{\partial}{\partial y}(\sin y) = \cos y = \frac{\partial}{\partial t}(t\cos y - y)$$

this ODE is exact.

$$\begin{aligned}
\psi(t, y) &= \int \sin y\,dt \\
&= t\sin y + h(y)
\end{aligned}$$

where $h(y)$ is an arbitrary function of y.

$$\begin{aligned}
\frac{\partial}{\partial y}\psi(t, y) &= \frac{\partial}{\partial y}(t\sin y + h(y)) \\
t\cos y + h'(y) &= t\cos y - y \\
h'(y) &= -y \\
h(y) &= -\frac{1}{2}y^2
\end{aligned}$$

Thus the solution to the exact ODE can be written implicitly as

$$t\sin y - \frac{1}{2}y^2 = C$$

where C is a constant.

3. (8 points) Let $y(t) = c_1 t^{-2} + c_2 t^{-2} \ln t$ and show that y solves the ODE below.

$$t^2 y'' + 5ty' + 4y = 0.$$

$$\begin{aligned} y(t) &= c_1 t^{-2} + c_2 t^{-2} \ln t \\ y'(t) &= (c_2 - 2c_1) t^{-3} - 2c_2 t^{-3} \ln t \\ y''(t) &= (-5c_2 + 6c_1) t^{-4} + 6c_2 t^{-4} \ln t \end{aligned}$$

Substituting into the ODE yields.

$$\begin{aligned} t^2 y'' + 5ty' + 4y &= t^2 \left[(-5c_2 + 6c_1) t^{-4} + 6c_2 t^{-4} \ln t \right] \\ &\quad + 5t \left[(c_2 - 2c_1) t^{-3} - 2c_2 t^{-3} \ln t \right] \\ &\quad + 4 \left[c_1 t^{-2} + c_2 t^{-2} \ln t \right] \\ &= \left[(-5c_2 + 6c_1) t^{-2} + 6c_2 t^{-2} \ln t \right] \\ &\quad + \left[(5c_2 - 10c_1) t^{-2} - 10c_2 t^{-2} \ln t \right] \\ &\quad + \left[4c_1 t^{-2} + 4c_2 t^{-2} \ln t \right] \\ &= c_1 [6 - 10 + 4] t^{-2} + c_2 [-5 + 5] t^{-2} + c_2 [6 - 10 + 4] t^{-2} \ln t \\ &= 0 \end{aligned}$$

4. (8 points each) Suppose the total mass of a boat and its passenger is m, the velocity of the boat is $v(t)$ and quantities P, S, and B are positive constants. Newton's Second Law of Motion models the motion of the boat as

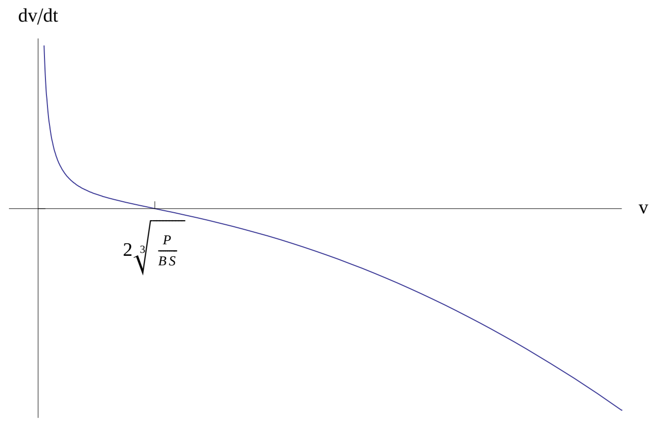
$$m \frac{dv}{dt} = \frac{8P}{v} - BSv^2.$$

(a) Determine all the equilibria for this ODE.
At equilibrium $\frac{dv}{dt} = 0$, so

$$\begin{aligned} 0 &= \frac{8P}{v_e} - BSv_e^2 \\ &= 8P - BSv_e^3 \\ v_e &= 2\sqrt[3]{\frac{P}{BS}}. \end{aligned}$$

(b) Classify each equilibrium as stable or unstable.

If we plot $\frac{dv}{dt}$ as a function of v, we see the following general shape to the graph.



Thus when $v < v_e$ we have $\frac{dv}{dt} > 0$ and when $v > v_e$ we have $\frac{dv}{dt} < 0$. This implies v_e is a stable equilibrium.

5. (12 points) Show that the following ODE is not exact and find an integrating factor depending on a single variable that makes the equation exact. You do not need to solve the resulting exact equation.

$$(2t + 2ty^2)\, dt + (t^2y + 2y + 3y^3)\, dy = 0$$

We can see that the ODE is not exact since

$$\frac{\partial}{\partial y}(2t + 2ty^2) = 4ty \neq 2ty = \frac{\partial}{\partial t}(t^2y + 2y + 3y^3).$$

If we look for an integrating factor of the form $\mu(y)$ then the following equation must be satisfied.

$$\begin{aligned}
\frac{\partial}{\partial y}[\mu(y)(2t + 2ty^2)] &= \frac{\partial}{\partial t}[\mu(y)(t^2y + 2y + 3y^3)] \\
\mu'(y)(2t + 2ty^2) + 4ty\mu(y) &= 2ty\mu(y) \\
\mu'(y)(1 + y^2) + y\mu(y) &= 0 \\
\mu'(y) + \frac{y}{1 + y^2}\mu(y) &= 0 \\
\frac{d}{dy}\left(\sqrt{1 + y^2}\mu(y)\right) &= 0 \\
\sqrt{1 + y^2}\mu(y) &= C \\
\mu(y) &= \frac{1}{\sqrt{1 + y^2}}
\end{aligned}$$

where we have chosen $C = 1$.

6. (12 points) Consider the second order linear constant coefficient homogeneous ODE

$$y'' - 2y' + y = 0.$$

Show that $\{e^t,\, te^t\}$ is a fundamental set of solutions to this ODE.

We can see that $y = e^t$ is a solution since

$$y'' - 2y' + y = e^t - 2e^t + e^t = 0.$$

Also, we can see that $y = te^t$ is a solution to the ODE because

$$y'' - 2y' + y = (2+t)e^t - 2(1+t)e^t + te^t = 0.$$

The two solutions form a fundamental set of solutions since

$$\begin{aligned} W &= \begin{vmatrix} e^t & te^t \\ e^t & (1+t)e^t \end{vmatrix} \\ &= (1+t)e^{2t} - te^{2t} \\ &= e^{2t} \\ &\neq 0 \end{aligned}$$

when $t = 0$.