Millersville University **Name** **Answer Key**
Mathematics Department
MATH 365, *Ordinary Differential Equations*, Test 2
October 29, 2008, 2:00-2:50PM

Please answer the following questions. Show all work and write neatly. Answers without justifying work will receive no credit. Partial credit will be given as appropriate, do not leave any problem blank. The point values of problems are indicated in parentheses.

1. Consider the second order linear ordinary differential equation,

$$y'' - \frac{2}{t+1}y' + \frac{2}{(t+1)^2}y = 0.$$

   (a) (4 points) Verify that $y_1(t) = t + 1$ is a solution to the ODE.

$$\begin{aligned}(t+1)'' - \frac{2}{t+1}(t+1)' + \frac{2}{(t+1)^2}(t+1) &= (0) - \frac{2}{t+1}(1) + \frac{2}{(t+1)} \\ &= 0\end{aligned}$$

   Thus $y_1(t) = t + 1$ is a solution to the ODE.

   (b) (8 points) Using reduction of order, find a second linearly independent solution to this ODE.

   We will assume that $y_2(t) = v(t)y_1(t) = v(t)(t + 1)$ is a solution to the ODE.

$$\begin{aligned}0 &= (v(t)(t+1))'' - \frac{2}{t+1}(v(t)(t+1))' + \frac{2}{(t+1)^2}(v(t)(t+1)) \\ &= v''(t)(t+1) + 2v'(t) - \frac{2}{t+1}(v'(t)(t+1) + v(t)) + \frac{2}{t+1}v(t) \\ &= v''(t)(t+1) + 2v'(t) - 2v'(t) - \frac{2}{t+1}v(t) + \frac{2}{t+1}v(t) \\ &= v''(t)(t+1) \\ 0 &= v''(t)\end{aligned}$$

   Thus $v(t) = At$ and $y_2 = t(t + 1) = t^2 + t$.

2. (10 points each) Find the general solutions to the following ordinary differential equations.

   (a) $2y'' - 5y' + 2y = 0$
   The characteristic equation is

$$\begin{aligned} 2r^2 - 5r + 2 &= 0 \\ (2r-1)(r-2) &= 0 \end{aligned}$$

   which implies $r_1 = 1/2$ and $r_2 = 2$. Thus the general solution to the ODE is

$$y(t) = c_1 e^{t/2} + c_2 e^{2t}.$$

   (b) $y'' - 6y' + 13y = 3e^{-2t}$
   The characteristic equation is

$$\begin{aligned} r^2 - 6r + 13 &= 0 \\ r &= \frac{6 \pm \sqrt{36 - 4(1)(13)}}{2} \\ &= 3 \pm 2i \end{aligned}$$

   which implies the complementary solution is

$$y_c(t) = e^{3t}(c_1 \cos 2t + c_2 \sin 2t).$$

   We may find the particular solution using the method of undetermined coefficients. Assume that $Y(t) = Ae^{-2t}$ then

$$\begin{aligned} 4Ae^{-2t} - 6(-2)Ae^{-2t} + 13Ae^{-2t} &= 3e^{-2t} \\ 4A + 12A + 13A &= 3 \\ 29A &= 3 \\ A &= \frac{3}{29}. \end{aligned}$$

   Thus the general solution to the ODE is

$$y(t) = e^{3t}(c_1 \cos 2t + c_2 \sin 2t) + \frac{3}{29}e^{-2t}.$$

(c) $y'' + 2y' + y = 0$
The characteristic equation is

$$\begin{aligned} r^2 + 2r + 1 &= 0 \\ (r+1)^2 &= 0 \end{aligned}$$

thus the roots are $r_1 = r_2 = -1$. The general solution to the ODE is

$$y(t) = c_1 e^{-t} + c_2 t e^{-t}.$$

(d) $y'' - 2y' + y = \ln t$
The characteristic equation is

$$\begin{aligned} r^2 - 2r + 1 &= 0 \\ (r-1)^2 &= 0 \end{aligned}$$

thus the roots are $r_1 = r_2 = 1$. The complementary solution to the ODE is

$$y(t) = c_1 e^{t} + c_2 t e^{t}.$$

We may find the particular solution using the method of variation of parameters. Note that $W(y_1, y_2)(t) = e^{2t}$ and thus

$$\begin{aligned} \mu_1'(t) &= -\frac{(\ln t)te^t}{e^{2t}} \\ &= -(\ln t)te^{-t} \\ \mu_1(t) &= -\int_1^t (\ln s)se^{-s}\,ds \end{aligned}$$

and

$$\begin{aligned} \mu_2'(t) &= \frac{(\ln t)e^t}{e^{2t}} \\ &= (\ln t)e^{-t} \\ \mu_2(t) &= \int_1^t (\ln s)e^{-s}\,ds. \end{aligned}$$

Therefore the general solution to the ODE is

$$y(t) = c_1 e^t + c_2 te^t - e^t \int_1^t (\ln s)se^{-s}\,ds + te^t \int_1^t (\ln s)e^{-s}\,ds.$$

3. Suppose that an object of mass $m = 1$ kg is attached to a spring with spring constant $k = 100$ N/m. The coefficient of damping in the spring/mass system is $\gamma = 6$ N/m/sec. The mass is set into motion from its equilibrium position with an upward velocity of 2 m/sec.

   (a) (6 points) Write down the initial value problem describing the displacement of the mass.

   Let $u(t)$ denote the displacement of the mass from its equilibrium position.

$$\begin{aligned} u'' + 6u' + 100u &= 0 \\ u(0) &= 0 \\ u'(0) &= -2 \end{aligned}$$

   (b) (10 points) Solve the initial value problem just found.

   The characteristic equation is

$$\begin{aligned} r^2 + 6r + 100 &= 0 \\ r &= \frac{-6 \pm \sqrt{36 - 4(1)(100)}}{2} \\ &= -3 \pm \sqrt{91}i. \end{aligned}$$

   Thus the general form of the solution is

$$u(t) = e^{-3t}(c_1 \cos\sqrt{91}t + c_2 \sin\sqrt{91}t).$$

   Applying the initial conditions we find

$$u(0) = 0 = c_1$$

   and

$$\begin{aligned} u'(t) &= e^{-3t}(-3c_2 \sin\sqrt{91}t + c_2\sqrt{91}\cos\sqrt{91}t) \\ u'(0) &= -2 = c_2\sqrt{91} \\ c_2 &= -\frac{2}{\sqrt{91}}. \end{aligned}$$

   Thus the solution to the IVP is

$$u(t) = -\frac{2}{\sqrt{91}} e^{-3t} \sin\sqrt{91}t$$

(c) (4 points) What is the quasi–frequency of the spring/mass system?

The quasi-frequency is

$$\mu = \frac{\sqrt{4mk - \gamma^2}}{2m} = \sqrt{91}.$$

(d) (4 points) If the spring/mass system is modified so that there is no damping present and then subjected to an external force of the form $F_0 \cos \omega t$, for what value of ω will resonance occur?

If $\gamma = 0$ then resonance occurs when $\omega = \omega_0$, in other words when

$$\omega = \sqrt{\frac{100}{1}} = 10.$$

4. (8 points) Determine whether the following pair of functions is linearly independent for $-\infty < t < \infty$.

$$y_1(t) = \sin^2 t, \quad y_2(t) = 1 - \cos 2t$$

According to the Wronskian

$$\begin{aligned}
W(y_1, y_2)(t) &= \begin{vmatrix} \sin^2 t & 1 - \cos 2t \\ 2\sin t \cos t & 2\sin 2t \end{vmatrix} \\
&= 2\sin^2 t \sin 2t - (1 - \cos 2t) 2 \sin t \cos t \\
&= 2\sin^2 t \sin 2t - (1 - \cos 2t) \sin 2t \\
&= (2\sin^2 t - 1 + \cos 2t) \sin 2t \\
&= (1 - \cos 2t - 1 + \cos 2t) \sin 2t \\
&= 0,
\end{aligned}$$

which implies the functions are linearly dependent for $-\infty < t < \infty$.

5. (8 points) Solve the following ODE by means of a power series solution about $t_0 = 0$. You must state the recurrence relation and the first four terms in each of two linearly independent solutions.

$$(2 + t^2)y'' - ty' + 4y = 0$$

If we assume that $y(t) = \sum_{n=0}^{\infty} a_n t^n$, differentiate the solution and substitute into the ODE we obtain

$$\begin{aligned}
0 &= (2 + t^2)\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - t\sum_{n=1}^{\infty} n a_n t^{n-1} + 4\sum_{n=0}^{\infty} a_n t^n \\
&= \sum_{n=2}^{\infty} 2n(n-1)a_n t^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n t^n - \sum_{n=1}^{\infty} n a_n t^n + \sum_{n=0}^{\infty} 4a_n t^n \\
&= \sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2} t^n + \sum_{n=0}^{\infty} n(n-1)a_n t^n - \sum_{n=0}^{\infty} n a_n t^n + \sum_{n=0}^{\infty} 4a_n t^n \\
&= \sum_{n=0}^{\infty} \big(2(n+2)(n+1)a_{n+2} + (n^2 - 2n + 4)a_n\big)\, t^n \\
a_{n+2} &= -\frac{n^2 - 2n + 4}{2(n+2)(n+1)} a_n
\end{aligned}$$

as the recurrence relation.

Assuming $a_0 = 1$ and $a_1 = 0$ then according to the recurrence relation $0 = a_{2m+1}$.

$$\begin{aligned}
a_2 &= -\frac{4}{4}a_0 = -1 \\
a_4 &= -\frac{4}{24}a_2 = \frac{1}{6} \\
a_6 &= -\frac{16 - 8 + 4}{2(6)(5)}a_4 = -\frac{1}{30}
\end{aligned}$$

Hence one solution is $y_1(t) = 1 - t^2 + \frac{1}{6}t^4 - \frac{1}{30}t^6 + \cdots$. Assuming $a_0 = 0$ and $a_1 = 1$ then according to the recurrence relation $0 = a_{2m}$.

$$\begin{aligned}
a_3 &= -\frac{1 - 2 + 4}{2(3)(2)}a_1 = -\frac{1}{4} \\
a_5 &= -\frac{9 - 6 + 4}{2(5)(4)}a_3 = \frac{7}{160} \\
a_7 &= -\frac{25 - 10 + 4}{2(7)(6)}a_5 = -\frac{19}{1920}
\end{aligned}$$

Hence another solution is $y_2(t) = t - \frac{1}{4}t^3 + \frac{7}{160}t^5 - \frac{19}{1920}t^7 + \cdots$.

6. (8 points) Show that $t_0 = \pi/2$ is an ordinary point for the following ODE. If $y(t) = \sum_{n=0}^{\infty} a_n \left(t - \frac{\pi}{2}\right)^n$ is a power series solution to the ODE, what is a lower bound for the radius of convergence of the series?

$$(\sin t)y'' - t\,y' + 2y = 0$$

The value $t_0 = \pi/2$ is an ordinary point for the ODE since $\sin \pi/2 = 1 \neq 0$. The radius of convergence of the power series is at least as large as the minimum distance from $t_0 = \pi/2$ to any solution of $\sin t = 0$. This implies a lower bound for the radius of convergence is $\rho = \pi/2$.