Millersville University **Name** **Answer Key**
Mathematics Department
MATH 365, *Ordinary Differential Equations*, Test 3
November 24, 2008, 2:00-2:50PM

Please answer the following questions. Show all work and write neatly. Answers without justifying work will receive no credit. Partial credit will be given as appropriate, do not leave any problem blank. The point values of problems are indicated in parentheses.

1. (10 points) The Chebyshev equation is

$$(1-x^2)y'' - xy' + \alpha^2 y = 0$$

where α is a constant. Show that $x = 1$ is a regular singular point and find the exponents of singularity.

We will let $P(x) = 1 - x^2$, $Q(x) = -x$, and $R(x) = \alpha^2$. Since $P(1) = 0$ then $x = 1$ is a singular point. We also note that

$$\begin{aligned}
\lim_{x\to 1}(x-1)\left(\frac{Q(x)}{P(x)}\right) &= \lim_{x\to 1}(x-1)\left(\frac{-x}{1-x^2}\right) = \lim_{x\to 1}\frac{x}{1+x} = \frac{1}{2} = p_0, \\
\lim_{x\to 1}(x-1)^2\left(\frac{R(x)}{P(x)}\right) &= \lim_{x\to 1}(x-1)^2\left(\frac{\alpha^2}{1-x^2}\right) = \lim_{x\to 1}\frac{-\alpha^2(x-1)}{1+x} = 0 = q_0.
\end{aligned}$$

Since the preceding two limits are finite, then $x = 1$ is a regular singular point. The exponents of singularity are the roots of the indicial equation:

$$\begin{aligned}
r(r-1) + p_0 r + q_0 &= 0 \\
r(r-1) + \frac{1}{2}r &= 0 \\
r\left(r - \frac{1}{2}\right) &= 0
\end{aligned}$$

$r_1 = \dfrac{1}{2}$ and $r_2 = 0$.

2. (15 points each) Solve the following ODEs/IVPs.

(a) $t^2y'' + 3ty' + 5y = 0$

If $z = \ln t$ then

$$\frac{d^2y}{dz^2} + 2\frac{dy}{dz} + 5y = 0$$

and the characteristic equation is

$$\begin{aligned} r^2 + 2r + 5y &= 0 \\ r &= -1 \pm 2i \end{aligned}$$

where $i = \sqrt{-1}$. Thus the general solution is

$$y(t) = t^{-1}\left(c_1 \cos(2\ln t) + c_2 \sin(2\ln t)\right).$$

(b) $2t^2y'' - 6ty' + 8y = 0$

The Euler equation above is equivalent to

$$t^2y'' - 3ty' + 4y = 0.$$

If $z = \ln t$ then

$$\frac{d^2y}{dz^2} - 4\frac{dy}{dz} + 4y = 0$$

and the characteristic equation is

$$\begin{aligned} r^2 - 4r + 4y &= 0 \\ r &= 2. \end{aligned}$$

Thus the general solution is

$$y(t) = t^2\left(c_1 + c_2 \ln t\right).$$

(c) $t^2y'' + 2ty' + \frac{17}{4}y = 0$; $y(1) = 2$, $y'(1) = -3$
If $z = \ln t$ then

$$\frac{d^2y}{dz^2} + \frac{dy}{dz} + \frac{17}{4}y = 0$$

and the characteristic equation is

$$\begin{aligned} r^2 + r + \frac{17}{4}y &= 0 \\ r &= -\frac{1}{2} \pm 2i \end{aligned}$$

where $i = \sqrt{-1}$. Thus the general solution is

$$y(t) = t^{-1/2}\left(c_1 \cos(2\ln t) + c_2 \sin(2\ln t)\right).$$

Applying the initial conditions we see that

$$\begin{aligned} y(1) &= 2 = c_1 \\ y'(1) &= -3 = -1 + 2c_2 \quad \Longrightarrow \quad c_2 = -1. \end{aligned}$$

Thus the solution to the initial value problem is

$$y(t) = t^{-1/2}\left(2\cos(2\ln t) - \sin(2\ln t)\right)$$

3. (15 points) Consider the ODE

$$xy'' + 2y = 0$$

which has a regular singular point at $x = 0$. Find the first three nonzero coefficients of a solution to the ODE corresponding to the larger of the two exponents of singularity.

Assuming $y(x) = \sum_{n=0}^{\infty} a_n x^{r+n}$.

$$\begin{aligned}
0 &= x\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{r+n-2} + 2\sum_{n=0} a_n x^{r+n} \\
&= \sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{r+n-1} + \sum_{n=1} 2a_{n-1}x^{r+n-1} \\
&= r(r-1)a_0 x^{r-1} + \sum_{n=1}^{\infty}(r+n)(r+n-1)a_n x^{r+n-1} + \sum_{n=1} 2a_{n-1}x^{r+n-1} \\
&= r(r-1)a_0 x^{r-1} + \sum_{n=1}^{\infty}\left[(r+n)(r+n-1)a_n + 2a_{n-1}\right]x^{r+n-1}
\end{aligned}$$

Thus the exponents of singularity are $r_1 = 1$ and $r_2 = 0$.

Now we will look for the first three nonzero coefficients of the series solution corresponding to $r = r_1 = 1$. The recurrence relation is

$$a_n = -\frac{2a_{n-1}}{n(n+1)}.$$

If $a_0 \neq 0$ is arbitrary then

$$\begin{aligned}
a_1 &= -\frac{2}{(2)(1)}a_0 \\
a_2 &= -\frac{2}{(3)(2)}a_1 = \frac{2^2}{3!2!}a_0 \\
a_3 &= -\frac{2}{(4)(3)}a_2 = -\frac{2^3}{4!3!}a_0 \\
&\vdots \\
a_n &= \frac{(-2)^n}{(n+1)!\,n!}a_0.
\end{aligned}$$

Thus one solution to the ODE has the form

$$y_1(x) = a_0 x\left[1 + \sum_{n=1}^{\infty}\frac{(-2)^n x^n}{(n+1)!\,n!}\right].$$

4. (15 points) Use the definition of the Laplace transform to find $\mathcal{L}\left\{t^2 e^{at}\right\}$, where a is a constant.

$$
\begin{aligned}
\mathcal{L}\left\{t^2 e^{at}\right\} &= \int_0^\infty e^{-st} t^2 e^{at}\, dt \\
&= \lim_{M\to\infty} \int_0^M t^2 e^{-(s-a)t}\, dt \\
&= \lim_{M\to\infty} \left[-\frac{t^2}{s-a} e^{-(s-a)t}\Big|_0^M - \frac{2t}{(s-a)^2} e^{-(s-a)t}\Big|_0^M - \frac{2}{(s-a)^3} e^{-(s-a)t}\Big|_0^M \right] \\
&= \lim_{M\to\infty} \left[-\frac{M^2}{s-a} e^{-(s-a)M} - \frac{2M}{(s-a)^2} e^{-(s-a)M} - \frac{2}{(s-a)^3} e^{-(s-a)M} + \frac{2}{(s-a)^3} \right] \\
&= \frac{2}{(s-a)^3}
\end{aligned}
$$

for $s > a$.

5. (15 points) Use the Laplace transform to solve the following IVP.

$$\begin{aligned} y'' + 3y' + 2y &= 0 \\ y(0) &= 0 \\ y'(0) &= 1 \end{aligned}$$

$$\begin{aligned} \mathcal{L}\left\{y'' + 3y' + 2y\right\} &= \mathcal{L}\left\{0\right\} \\ s^2Y(s) - sy(0) - y'(0) + 3(sY(s) - y(0)) + 2Y(s) &= 0 \\ (s^2 + 3s + 2)Y(s) - 1 &= 0 \\ Y(s) &= \frac{1}{s^2 + 3s + 2} \\ &= \frac{1}{(s+1)(s+2)} \\ &= \frac{1}{s+1} - \frac{1}{s+2} \\ y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} \\ &= e^{-t} - e^{-2t} \end{aligned}$$