Millersville University Name Answer Key

Department of Mathematics

MATH 365, *Ordinary Differential Equations*, **Final Examination**

April 29, 2004, 2:45PM-4:45PM

Please answer the following questions. Show all work and write neatly. Answers without justifying work will receive no credit. Partial credit will be given as appropriate, do not leave any problem blank. The point values of problems are indicated in parentheses.

1. (5 points each) Solve the following first order differential equations and initial value problems.

(a) $y' = e^{2t} + 3y$

$y' - 3y = e^{2t}$ (1st order, linear)

integrating factor: $\mu(t) = e^{-3t}$

$$\frac{d}{dt}\left(ye^{-3t}\right) = e^{-t}$$

$$ye^{-3t} = -e^{-t} + C$$

$$y(t) = -e^{2t} + Ce^{3t}$$

(b) $y' = \dfrac{t^2-1}{y^2+1}, \qquad y(-1) = 1$ (separable)

$$(y^2+1)\,dy = (t^2-1)\,dt$$

$$\int_1^y (s^2+1)\,ds = \int_{-1}^t (s^2-1)\,ds$$

$$\frac{1}{3}y^3 + y - \frac{4}{3} = \frac{1}{3}t^3 - t - \left(\frac{-1}{3} - (-1)\right)$$

$$\frac{1}{3}y^3 + y = \frac{1}{3}t^3 - t + 2/3$$

(c) $(x+y)dx + (x+2y)dy = 0, \qquad y(2) = 3$

$$\frac{\partial}{\partial y}(x+y) = 1 = \frac{\partial}{\partial x}(x+2y) \quad \text{(exact equation)}$$

$$\psi(x,y) = \int (x+y)\,dx = \frac{x^2}{2} + xy + h(y)$$

$$\frac{\partial \psi}{\partial y} = x + h'(y) = x + 2y \Rightarrow h'(y) = 2y \Rightarrow h(y) = y^2$$

Thus $\frac{x^2}{2} + xy + y^2 = C = \frac{2^2}{2} + 2(3) + 3^2 = 17$

$$\frac{x^2}{2} + xy + y^2 = 17$$

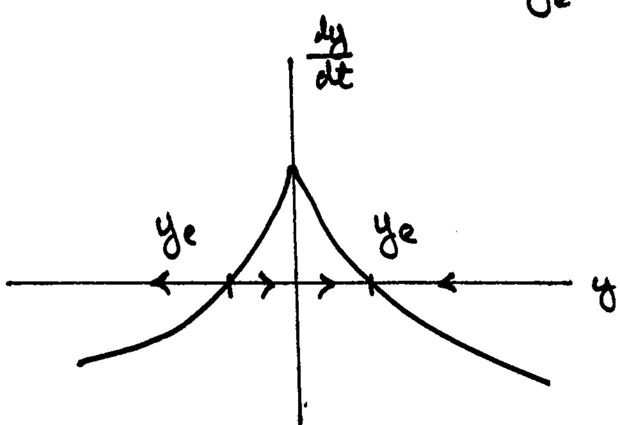
2. (5 points each) For each of the following ordinary differential equations find the equilibrium points and determine whether they are stable or unstable.

(a) $y' = k - \alpha\pi\left(\frac{3a}{\pi h}\right)^{2/3} y^{2/3}$, constants k, α, a, and h are all positive.

At equilibrium $y' = 0 = k - \alpha\pi\left(\frac{3a}{\pi h}\right)^{2/3} y^{2/3}$

$$y^{2/3} = \frac{k}{\alpha\pi}\left(\frac{\pi h}{3a}\right)^{2/3}$$

$$y_e = \pm\left(\frac{k}{\alpha\pi}\right)^{3/2}\frac{\pi h}{3a}$$



$y = -\left(\frac{k}{\alpha\pi}\right)^{3/2}\frac{\pi h}{3a}$ is unstable

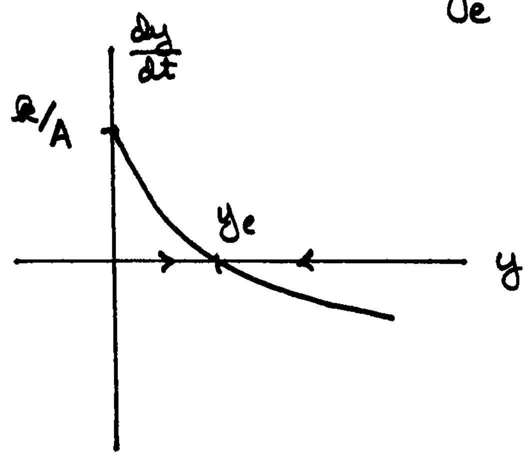
$y = \left(\frac{k}{\alpha\pi}\right)^{3/2}\frac{\pi h}{3a}$ is stable

(b) $y' = (k - \alpha a\sqrt{2gy})/A$, constants k, α, a, A, and g are all positive.

At equilibrium $y' = 0 = \frac{1}{A}\left(k - \alpha a\sqrt{2gy}\right)$

$$\sqrt{2gy} = \frac{k}{\alpha a}$$

$$y_e = \frac{1}{2g}\left(\frac{k}{\alpha a}\right)^2 \quad \text{stable}$$

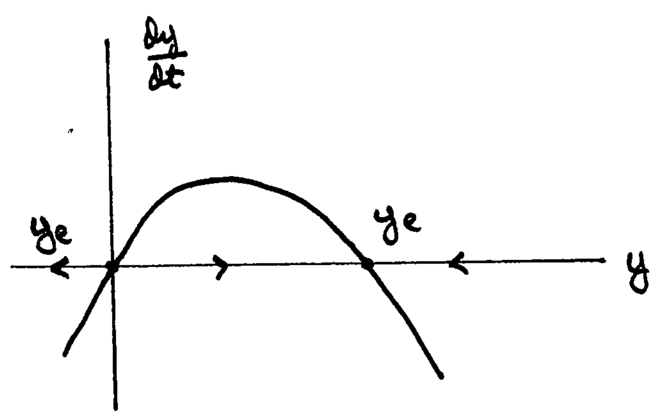


(c) $y' = r(1 - \dfrac{y}{K})y - Ey$, constants r, E, and K, are all positive with $E < r$.

At equilibrium, $y' = 0 = r\left(1 - \frac{y}{K}\right)y - Ey$

$$0 = y\left[r\left(1 - \frac{y}{K}\right) - E\right]$$

Thus $y_e = 0$ or $y_e = K\left(1 - E/r\right) > 0$



$y_e = 0$ is unstable.

$y_e = K\left(1 - E/r\right)$ is stable.

3. (5 points) The radioactive isotope of Strontium, ^{90}Sr decays at a rate proportional to the amount of ^{90}Sr present. Its half-life is $t_{1/2} = 28.5$ years. ^{90}Sr is produced during the detonation of nuclear weapons. In 2004 at the site of a nuclear test which took place in 1956, the concentration of ^{90}Sr is found to be 0.143 milligrams/m^2. What was the concentration of ^{90}Sr at the test site immediately after the nuclear detonation?

$$k = \frac{\ln 2}{t_{1/2}} = \frac{\ln 2}{28.5} = 0.02432 \text{ yr}^{-1}$$

$$y(t) = y(0)e^{-kt}$$

$$0.143 = y(0)e^{-k(48)}$$

$$y(0) = 0.143e^{k(48)} \approx 0.460 \text{ mg/m}^2$$

4. (8 points each) Solve the following second order linear differential equations and initial value problems.

(a) $y'' + y' + 4y = 0, \qquad y(0) = 1 \quad y'(0) = 0$

Characteristic equation: $r^2 + r + 4 = 0$

$$r = \frac{-1 \pm \sqrt{1 - 4(1)(4)}}{2}$$

$$r = \frac{-1}{2} \pm i\frac{\sqrt{15}}{2}$$

$$y(t) = e^{-t/2}\left(c_1 \cos\frac{\sqrt{15}}{2}t + c_2 \sin\frac{\sqrt{15}}{2}t\right)$$

$$y(0) = 1 = c_1$$

$$y'(t) = \frac{-1}{2}e^{-t/2}\left(\cos\frac{\sqrt{15}}{2}t + c_2 \sin\frac{\sqrt{15}}{2}t\right) + e^{-t/2}\left(-\frac{\sqrt{15}}{2}\sin\frac{\sqrt{15}}{2}t + c_2\frac{\sqrt{15}}{2}\cos\frac{\sqrt{15}}{2}t\right)$$

$$y'(0) = 0 = \frac{-1}{2} + c_2\frac{\sqrt{15}}{2} \Rightarrow c_2 = \frac{1}{\sqrt{15}}$$

Thus $y(t) = e^{-t/2}\left(\cos\frac{\sqrt{15}}{2}t + \frac{1}{\sqrt{15}}\sin\frac{\sqrt{15}}{2}t\right)$

(b) $y'' + 2y' + y = e^{-2t}$

Homogeneous solution: $y_h(t) = c_1 e^{-t} + c_2 t e^{-t}$.

Non-homogeneous solution: $Y(t) = Ae^{-2t}$

$4Ae^{-2t} - 4Ae^{-2t} + Ae^{-2t} = e^{-2t} \Rightarrow A = 1$

General solution: $y(t) = (c_1 + c_2 t)e^{-t} + e^{-2t}$

(c) $y'' - 9y = 0, \qquad y(0) = 0 \quad y'(0) = 1$

Characteristic equation: $r^2 - 9 = 0$

$r_1 = 3, \ r_2 = -3$

$y(t) = c_1 e^{3t} + c_2 e^{-3t}$

$y(0) = c_1 + c_2 = 0$

$y'(0) = 3c_1 - 3c_2 = 1$

$\Rightarrow c_1 = 1/6, \ c_2 = -1/6$

$y(t) = \frac{1}{6} e^{3t} - \frac{1}{6} e^{-3t}$

(d) $x^2y'' + 4xy' + 2y = 0$ Euler equation

Transformed equation: $\frac{d^2y}{dz^2} + 3\frac{dy}{dz} + 2y = 0$

Characteristic equation: $r^2 + 3r + 2 = 0$

$(r+2)(r+1) = 0$

$r_1 = -2,\ r_2 = -1$

$y(x) = c_1/x + c_2/x^2$

(e) $y'' + 4y = \sec^2 2t$

Homogeneous solution: $y_h(t) = c_1\cos 2t + c_2 \sin 2t$

Use variation of parameters to find the non-homogeneous solution.

$$W(t) = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2$$

$$w_1'(t) = \frac{-\sec^2 2t \sin 2t}{2} = \frac{-1}{2}\sec 2t \tan 2t$$

$$w_1(t) = \frac{-1}{4}\sec 2t$$

$$w_2'(t) = \frac{\sec^2 2t \cos 2t}{2} = \frac{1}{2}\sec 2t$$

$$w_2(t) = \frac{1}{4}\ln|\sec 2t + \tan 2t|$$

General solution:

$$y(t) = c_1\cos 2t + c_2\sin 2t - \frac{1}{4} + \frac{1}{4}\sin 2t \ln|\sec 2t + \tan 2t|$$

5. (5 points) Newton's Law of Cooling states that the rate of change of temperature of an object is proportional to the difference between the temperatures of the object and its environment. If the temperature of a pizza straight from the oven is 350°F and the environment has a temperature of 80°F, write down an initial value problem for the temperature T, of the pizza as a function of time t. Please state clear any assumptions you make about the algebraic sign (positive or negative) of the proportionality constant.

$$\frac{dT}{dt} = -k(T-80) \qquad \text{if } k>0$$

$$T(0) = 350$$

6. (5 points each) Consider this example of the Laguerre differential equation,

$$xy'' + (1-x)y' + y = 0.$$

(a) Show that $x_0 = 0$ is a regular singular point for this ordinary differential equation.

Let $P(x) = x$, then $P(0) = 0 \Rightarrow x_0 = 0$ is a singular point.

$$\lim_{x \to 0} x \frac{1-x}{x} = \lim_{x\to 0}(1-x) = 1 = p_0 < \infty$$

$$\lim_{x\to 0} x^2 \frac{1}{x} = 0 = q_0 < \infty$$

Thus $x_0 = 0$ is a regular singular point.

(b) Using power series techniques, find a solution $y_1(x)$ to the Laguerre equation. State the first four terms in the power series solution (unless the series terminates sooner).

Indicial equation: $r(r-1)+r=0$

$$r^2=0 \implies r_1=r_2=0$$

Assume $y(x)=\sum\limits_{n=0}^{\infty} a_n x^n$.

$$x\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + (1-x)\sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n^2 a_n x^{n-1} + \sum_{n=0}^{\infty} (1-n)a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1)^2 a_{n+1} x^n + \sum_{n=0}^{\infty} (1-n)a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1)^2 a_{n+1} + (1-n)a_n\right] x^n = 0$$

Recurrence relation: $a_{n+1} = \dfrac{n-1}{(n+1)^2} a_n$ for $n=0,1,\ldots$

Let $a_0 = 1$

$a_1 = \dfrac{-1}{1^2} a_0 = -1$

$a_2 = 0$

$a_3 = a_4 = \cdots = 0$

Thus $y_1(x) = 1-x$.

(c) Using reduction of order, set up a first order linear ordinary differential equation which can be used to find a second linearly independent solution $y_2(x)$ to the Laguerre equation. You do not need to solve this differential equation.

Let $y_2(x) = v(x)(1-x)$

$y_2'(x) = v'(x)(1-x) - v(x)$

$y_2''(x) = v''(x)(1-x) - 2v'(x)$

Thus we have:

$$x\left[v''(x)(1-x) - 2v'(x)\right] + (1-x)\left[v'(x)(1-x) - v(x)\right] + v(x)(1-x) = 0$$

$$v''(x)\left(x - x^2\right) + v'(x)\left(1 - 4x + x^2\right) = 0$$

$$v''(x) + \frac{1-4x+x^2}{x-x^2}\, v'(x) = 0$$

Let $w(x) = v'(x)$ then

$$w'(x) + \left(-1 + \frac{2x-1}{x^2-x} + \frac{1}{x-1}\right) w(x) = 0$$

We can integrate this 1st order linear ODE using the integrating factor $\mu(x) = x(x-1)^2 e^{-x}$.

7. (5 points) A mass of 10 kg is stretches a spring 3.5 cm. The motion of the spring/mass system is damped by a frictional force with damping coefficient γ. Determine the value of γ for which the system is critically damped. State the units of measurement used in the value of γ you find.

$$mg = kL$$

$$(10)(9.8) = k(0.035) \Rightarrow k = \frac{(10)(9.8)}{0.035} = 2800 \text{ N/m}$$

$$mu'' + \gamma u' + ku = 0$$

Characteristic equation: $10r^2 + \gamma r + 2800 = 0$

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4(10)(2800)}}{20}$$

System is critically damped when

$$\gamma^2 = 112000$$

$$\gamma = 10\sqrt{1120} = 40\sqrt{70} \approx 334.664 \left(\frac{kg \cdot N}{m}\right)^{1/2}$$

$$\gamma = 40\sqrt{70}\left(\frac{kg^2\, m}{m \cdot s^2}\right) = 40\sqrt{70} \text{ kg/s}$$