

Please answer the following questions. Show all work and write neatly. Answers without justifying work will receive no credit. Partial credit will be given as appropriate, do not leave any problem blank. The point values of problems are indicated in parentheses.

1. (10 points) Consider the initial value problem

$$\begin{aligned} y' &= |y| \\ y(0) &= 0 \end{aligned}$$

Does there exist a unique solution to this initial value problem? You must justify your answer.

Let R be a rectangle containing the origin $(0,0)$ in its interior. If $f(t,y) = |y|$ then $f(t,y)$ is continuous on R . Since $\frac{\partial f}{\partial y}$ is undefined at $y=0$ the uniqueness theorem fails.

However if we consider the problem $y' = y$, $y(0) = 0$ then we see the unique solution is $y(t) \equiv 0$. For the problem $y' = -y$ with $y(0) = 0$ the unique solution is $y(t) \equiv 0$. Since $|y| = \begin{cases} y & \text{if } y \geq 0 \\ -y & \text{if } y < 0 \end{cases}$

then the unique solution to the IVP: $y' = |y|$, $y(0) = 0$ is $y(t) \equiv 0$.

2. (10 points each) Please find the solutions to the following ordinary differential equations and initial value problems.

First order linear IVP

(a) $y' + \frac{2}{t}y = \frac{\sin t}{t^2}$, $y(2) = 1$ $v(t) = e^{\int 2/t dt} = e^{2 \ln t} = t^2$

$$\frac{d}{dt}(y t^2) = \sin t$$

$$y t^2 = -\cos t + C$$

$$y(t) = \frac{C}{t^2} - \frac{\cos t}{t^2}$$

Since $y(2) = 1 = \frac{C}{4} - \frac{\cos 2}{4}$

$$4 = C - \cos 2$$

$$C = 4 + \cos 2$$

$$C \approx 3.5839$$

$$y(t) = \frac{4 + \cos 2 - \cos t}{t^2}$$

(b) $(e^x \sin y - 3y)dx - (3x - e^x \cos y)dy = 0, \quad y(0) = 0$

This equation is exact since, $\frac{\partial}{\partial y}(e^x \sin y - 3y)$
 $= e^x \cos y - 3$
 $= \frac{\partial}{\partial x}(-3x + e^x \cos y).$

Let $\psi(x, y) = \int (e^x \sin y - 3y) dx = e^x \sin y - 3xy + h(y).$

$\frac{\partial \psi}{\partial y} = e^x \cos y - 3x + h'(y) = -3x + e^x \cos y \Rightarrow h'(y) = 0$

Thus $h(y)$ is a constant.

Hence, $\psi(x, y) = e^x \sin y - 3xy = C$

$\psi(0, 0) = e^0 \sin(0) - 3(0) = 0 = C$

Finally $\psi(x, y) = e^x \sin y - 3xy = 0.$

(c) $y' = e^{t+y}$ This equation is separable.

$$\frac{dy}{dt} = e^t e^y$$

$$e^{-y} dy = e^t dt$$

$$-e^{-y} = e^t + C$$

$$e^{-y} = C - e^t$$

$$-y = \ln(C - e^t)$$

$$y(t) = -\ln(C - e^t)$$

$$(d) y' = ty^3(1+t^2)^{-1/2}, \quad y(0) = 1$$

$$\begin{aligned} \frac{dy}{dt} &= \frac{t}{\sqrt{1+t^2}} y^3 \\ y^{-3} dy &= \frac{t}{\sqrt{1+t^2}} dt \\ 2y^{-3} dy &= \frac{2t}{\sqrt{1+t^2}} dt \\ -y^{-2} &= 2\sqrt{1+t^2} + C \\ \frac{1}{y^2} &= C - 2\sqrt{1+t^2} \\ y^2(t) &= \frac{1}{C - 2\sqrt{1+t^2}} \\ y(t) &= \sqrt{\frac{1}{3 - 2\sqrt{1+t^2}}} \end{aligned}$$

$$(e) ty' + (t+1)y = 2te^{-t}$$

$$y' + \left(1 + \frac{1}{t}\right)y = 2e^{-t}$$

$$\frac{d}{dt} (te^t y) = 2t$$

$$te^t y = t^2 + C$$

$$y(t) = te^{-t} + \frac{C}{te^t}$$

This equation is separable.

Integrate both sides.

$$\begin{aligned} \text{since } y^2(0) &= 1^2 = \frac{1}{C-2} \\ C &= 3 \end{aligned}$$

This equation is first order linear.

$$\begin{aligned} w(t) &= e^{\int (1+1/t) dt} \\ &= e^{t + \ln t} \\ &= te^t \end{aligned}$$

3. (10 points) A hard-boiled egg is removed from a pot of water boiling at 212°F and placed on the kitchen table to cool. The temperature in the kitchen is 72°F . After five minutes the temperature of the egg is 160°F . At what time will be egg's temperature be 140°F ? Round your answer to the nearest whole minute.

Assuming the egg obeys Newton's Law of Cooling, then

$$\frac{dT}{dt} = -k(T - 72)$$

$$T(0) = 212$$

$$T' + kT = 72k \quad u(t) = e^{kt}$$

$$\frac{d}{dt} (Te^{kt}) = 72ke^{kt}$$

$$Te^{kt} = 72e^{kt} + C$$

$$T(t) = 72 + Ce^{-kt}$$

$$\text{since } T(0) = 212 = 72 + C$$

$$C = 140$$

$$\text{Thus } T(t) = 72 + 140e^{-kt}$$

$$T(5) = 160 = 72 + 140e^{-5k}$$

$$\frac{22}{35} = e^{-5k} \Rightarrow k = \frac{1}{5} \ln\left(\frac{35}{22}\right) \approx 0.0929 \text{ min}^{-1}$$

$$T(t) = 140 = 72 + 140e^{-kt}$$

$$\frac{17}{35} = e^{-kt}$$

$$t = \frac{1}{k} \ln\left(\frac{35}{17}\right) = \frac{5 \ln\left(\frac{35}{17}\right)}{\ln\left(\frac{35}{22}\right)} \approx 7.8 \text{ min.}$$

4. (6 points each) The Schaefer differential equation given below is used to model the population levels of a species which obeys the logistic equation for bounded growth and is subject to harvesting at a rate proportional to the size of the population. For example, salmon may obey the logistic equation for population growth, but people fish for salmon and thus also keep the salmon population from growing unbounded.

$$y' = ry \left(1 - \frac{y}{K} \right) - Ey$$

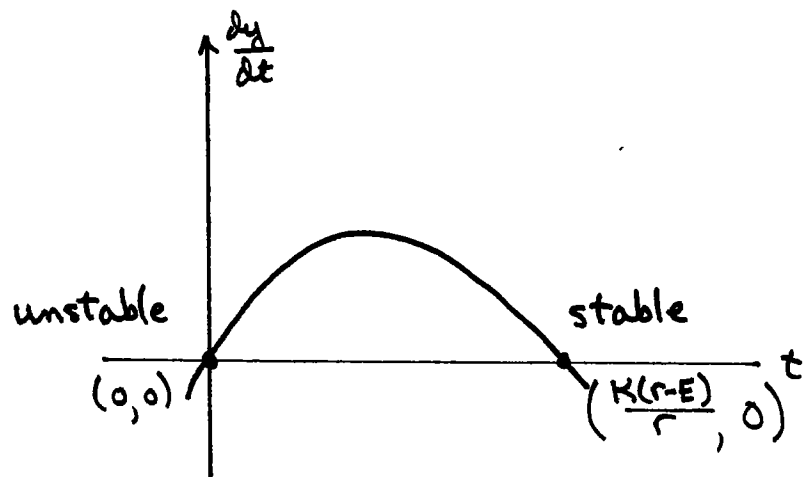
The constants are: $K > 0$ the carrying capacity of the environment, $r > 0$ the growth rate of the population, and $E > 0$ the effort put into harvesting.

- (a) Assuming $E < r$ find the equilibria for the model and determine the stability properties of the equilibria. You must justify your answer.

$$y' = (r-E)y - \frac{r}{K}y^2 = y \left[(r-E) - \frac{r}{K}y \right]$$

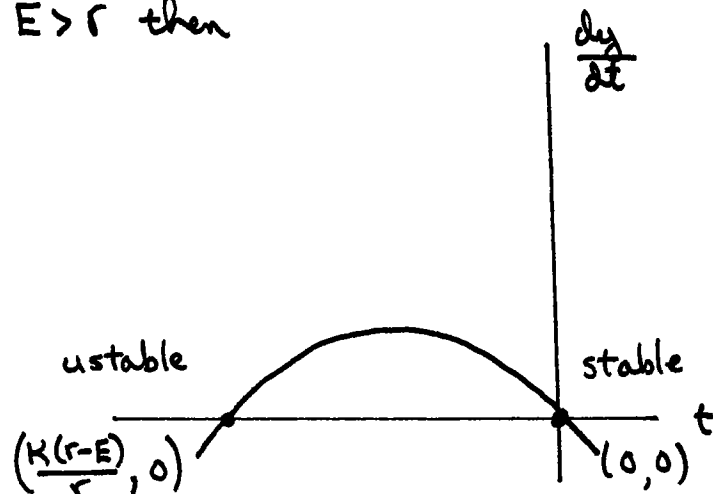
Thus the equilibria are $y=0$ and $y = \frac{r-E}{r/K} = \frac{K(r-E)}{r}$

If $E < r$ then



- (b) Assuming $E > r$ find the equilibria for the model and determine the stability properties of the equilibria. You must justify your answer.

If $E > r$ then



- (c) Since humans can control how much effort they put into fishing for salmon (for example), economically and environmentally, which is better $E < r$ or $E > r$? You must justify your answer.

Since if $E < r$ there is a stable positive equilibrium level for the population, this is the preferable situation. If $E > r$ only the $y=0$ population level is stable meaning the population will become extinct.

5. (12 points) Consider the initial value problem

$$y' = ty - t^2, \quad y(0) = 0.$$

Using the function $\phi_0(t) = 0$ as the initial Picard iterate, find the next three Picard iterates: $\phi_1(t)$, $\phi_2(t)$, and $\phi_3(t)$.

Let $f(t, y) = ty - t^2$

$$\phi_0(t) = 0$$

$$\phi_1(t) = \int_0^t f(s, \phi_0(s)) ds = \int_0^t s(0) - s^2 ds = -\frac{t^3}{3}$$

$$\phi_2(t) = \int_0^t s\phi_1(s) - s^2 ds = \int_0^t -\frac{s^4}{3} - s^2 ds = -\frac{t^5}{15} - \frac{t^3}{3}$$

$$\begin{aligned} \phi_3(t) &= \int_0^t s\phi_2(s) - s^2 ds = \int_0^t -\frac{s^6}{15} - \frac{s^4}{3} - s^3 ds \\ &= -\frac{t^7}{105} - \frac{t^5}{15} - \frac{t^3}{3} \end{aligned}$$