Millersville University
Mathematics Department
MATH 365, *Ordinary Differential Equations*, Test 2
March 5, 2004

Name Answer Key

Please answer the following questions. Show all work and write neatly. Answers without justifying work will receive no credit. Partial credit will be given as appropriate, do not leave any problem blank. The point values of problems are indicated in parentheses.

1. (10 points) Legendre's differential equation is

$$(1-t^2)y'' - 2ty' + \alpha(\alpha+1)y = 0,$$

where α is a real number. Without solving Legendre's equation, find the Wronskian of two linearly independent solutions to the equation.

$$W(t) = \exp\left(-\int \frac{-2t}{1-t^2}\,dt\right)$$
$$= \exp\left(-\ln|1-t^2| + c\right)$$
$$= \frac{C}{1-t^2}$$

2. Solve the following ordinary differential equations and initial boundary value problems.

(a) (11 points) $y'' + 3y' - 18y = 0; \qquad y(0) = 1, \quad y'(0) = 0$

Homogeneous equation: $y'' + 3y' - 18y = 0$

Characteristic equation: $r^2 + 3r - 18 = 0$

$(r+6)(r-3) = 0$

$r_1 = -6, \quad r_2 = 3$

General solution: $y(t) = c_1 e^{-6t} + c_2 e^{3t}$

$y(0) = 1 = c_1 + c_2$

$y'(t) = -6c_1 e^{-6t} + 3c_2 e^{3t}$

$y'(0) = 0 = -6c_1 + 3c_2$

$c_1 + c_2 = 1$

$-6c_1 + 3c_2 = 0$

$c_1 = 1/3 \quad c_2 = 2/3$

Thus $y(t) = \frac{1}{3}e^{-6t} + \frac{2}{3}e^{3t}$

(b) (15 points) $y'' + 2y = t + 3e^{2t}$

Homogeneous equation: $y'' + 2y = 0$

Homogeneous solution: $y_h(t) = c_1 \cos\sqrt{2}t + c_2 \sin\sqrt{2}t$

Particular solution: $Y(t) = At + B + Ce^{2t}$

$$4Ce^{2t} + 2At + 2B + 2Ce^{2t} = t + 3e^{2t}$$

$6C = 3 \Rightarrow C = 1/2$

$2A = 1 \Rightarrow A = 1/2$

$2B = 0 \Rightarrow B = 0$

Thus $Y(t) = \frac{1}{2}t + \frac{1}{2}e^{2t}$.

General solution: $y(t) = c_1 \cos\sqrt{2}t + c_2 \sin\sqrt{2}t + \frac{1}{2}t + \frac{1}{2}e^{2t}$

(c) (8 points) $4y'' - 12y' + 9y = 0$

Characteristic equation: $4r^2 - 12r + 9 = 0$

$$(2r-3)^2 = 0$$

$$r_1 = r_2 = 3/2$$

General solution: $y(t) = c_1 e^{3t/2} + c_2 t e^{3t/2}$

(d) (11 points) $y'' + 2y' + 5y = 0; \qquad y(0) = 0, \quad y'(0) = 1$

Characteristic equation: $r^2 + 2r + 5 = 0$

$$r = \frac{-2 \pm \sqrt{4 - 4(5)}}{2} = -1 \pm 2i$$

General solution: $y(t) = e^{-t}(c_1 \cos 2t + c_2 \sin 2t)$

$y(0) = 0 = c_1$

$y'(t) = -e^{-t} c_2 \sin 2t + e^{-t} 2c_2 \cos 2t$

$y'(0) = 1 = 2c_2 \Rightarrow c_2 = 1/2$

Thus $y(t) = \frac{1}{2} e^{-t} \sin 2t$

(e) (15 points) $y'' - 2y' + y = \dfrac{e^t}{1 + t^2}$

Homogeneous equation: $y'' - 2y' + y = 0$

Characteristic equation: $r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$r_1 = r_2 = 1$

Homogeneous solution: $y_h(t) = c_1 e^t + c_2 te^t$.

We must use variation of parameters to find the nonhomogeneous solution:

$$W(t) = \begin{vmatrix} e^t & te^t \\ e^t & e^t(t+1) \end{vmatrix} = e^{2t} \begin{vmatrix} 1 & t \\ 1 & t+1 \end{vmatrix} = e^{2t}$$

$$u_1'(t) = \frac{-\frac{e^t}{1+t^2} te^t}{e^{2t}} = \frac{-t}{1+t^2} \Rightarrow u_1(t) = -\frac{1}{2} \ln(1+t^2)$$

$$u_2'(t) = \frac{\frac{e^t}{1+t^2} e^t}{e^{2t}} = \frac{1}{1+t^2} \Rightarrow u_2(t) = \arctan t$$

Thus the general solution is

$$y(t) = c_1 e^t + c_2 te^t + -\frac{e^t}{2} \ln(1+t^2) + te^t \arctan t$$

3. (10 points) Are the functions $f(t) = \cos^2 t$ and $g(t) = 1 + \cos 2t$ linearly independent on the set of real numbers?

$$W(f,g)(t) = \begin{vmatrix} \cos^2 t & 1+\cos 2t \\ -2\cos t \sin t & -2\sin 2t \end{vmatrix}$$

$$= -2\cos^2 t \sin 2t + (1+\cos 2t)(2\cos t \sin t)$$

$$= -2\cos^3 t\,(2)\sin t + 2\cos t \sin t + 2\cos t \cos 2t \sin t$$

$$= -4\cos^3 t \sin t + 2\cos t \sin t + 2\cos t\,(\cos^2 t - \sin^2 t)\sin t$$

$$= -4\cos^3 t \sin t + 2\cos^3 t \sin t + 2\cos t \sin t - 2\cos t \sin^3 t$$

$$= -2\cos^3 t \sin t - 2\cos t \sin^3 t + 2\cos t \sin t$$

$$= -2\cos t \sin t\,(\cos^2 t + \sin^2 t) + 2\cos t \sin t$$

$$= -2\cos t \sin t + 2\cos t \sin t$$

$$= 0$$

Thus the functions are linearly dependent.

4. (10 points) Suppose $y(t)$ is any solution to the equation

$$y'' + (3-\alpha)y' - 2(\alpha - 1)y = 0.$$

Find the value of α for which $\lim_{t\to\infty} y(t) = 0$.

Characteristic equation:

$$r^2 + (3-\alpha)r - 2(\alpha-1) = 0$$

$$(r+2)(r-(\alpha-1)) = 0$$

$$r_1 = -2, \quad r_2 = \alpha - 1$$

General solution: $y(t) = c_1 e^{-2t} + c_2 e^{(\alpha-1)t}$.

Since $\lim_{t\to\infty} e^{-2t} = 0$ then $\lim_{t\to\infty} y(t) = 0$

iff $\lim_{t\to\infty} e^{(\alpha-1)t} = 0 \Rightarrow \alpha - 1 < 0 \Rightarrow \alpha < 1.$

5. (10 points) If $y_1(t) = \sin(t^2)$ solves the differential equation

$$ty'' - y' + 4t^3 y = 0,$$

then find a second linearly independent solution to the equation.

Let $y_2(t) = v(t)\sin(t^2)$

$$y_2'(t) = v'(t)\sin(t^2) + v(t)(2t)\cos(t^2)$$

$$y_2''(t) = v''(t)\sin(t^2) + 4t v'(t)\cos(t^2) + v(t)\left[2\cos(t^2) - 4t^2\sin(t^2)\right]$$

$$t\left[v''\sin(t^2) + 4tv'\cos(t^2) + 2v\cos(t^2) - 4t^2 v\sin(t^2)\right] - v'\sin(t^2) - 2t\,v\cos(t^2) + 4t^3 v\sin(t^2) = 0$$

$$tv''\sin(t^2) + \left[4t^2\cos(t^2) - \sin(t^2)\right]v' = 0$$

$$v'' + \left(4t\cot(t^2) - \frac{1}{t}\right)v' = 0$$

Let $w = v'$ and $w' = v''$, then

$$w + \left(4t\cot(t^2) - \frac{1}{t}\right)w = 0$$

integrating factor:

$$\mu(t) = e^{\int 4t\cot(t^2) - 1/t\,dt}$$

$$\mu(t) = e^{2\ln|\sin(t^2)| - \ln|t|} = \frac{\sin^2(t^2)}{t}$$

$$\frac{d}{dt}\left(w\,\frac{\sin^2(t^2)}{t}\right) = 0$$

$$w(t) = \frac{t}{\sin^2(t^2)} = t\csc^2(t^2)$$

$$v(t) = -\frac{1}{2}\cot(t^2)$$

Thus $y_2(t) = -\frac{1}{2}\cos(t^2)$.